## Online Appendix

This appendix includes all the missing proofs and the ancillary facts used in the main body of the paper. We begin with a section on facts instrumental for Theorem 1 and Proposition 5.

## Foundation

Recall the definition of $\geqslant^{\prime}$ in Section 5, that is,

$$
p \geqslant^{\prime} q \stackrel{\text { def }}{\Longleftrightarrow} \lambda p+(1-\lambda) r \geqslant \lambda q+(1-\lambda) r \quad \forall \lambda \in(0,1], \forall r \in \Delta .
$$

The goal of this section is to provide a Multi-Expected Utility representation for $\geqslant^{\prime}$.
Lemma 1. Let $\geqslant$ be a binary relation on $\Delta$ that satisfies Weak Order. The following statements are true:

1. The relation $\geqslant$ satisfies $M$-NCI if and only if for each $p \in \Delta$ and for each $m \in \mathbb{R}$

$$
p \geqslant \delta_{m e_{1}} \Longrightarrow p \geqslant^{\prime} \delta_{m e_{1}} . \quad \text { (Equivalently } p \not \not ㇒ ⿻^{\prime} \delta_{m e_{1}} \Longrightarrow \delta_{m e_{1}}>p \text {.) }
$$

2. If $\geqslant$ satisfies Monotonicity, then for each $x, y \in \mathbb{R}^{k}$

$$
\begin{equation*}
x>y \Longrightarrow \delta_{x}>^{\prime} \delta_{y} \tag{12}
\end{equation*}
$$

3. If $\geqslant$ satisfies Monetary equivalent, then for each $x, y \in \mathbb{R}^{k}$ there exists $m \in \mathbb{R}_{+}$such that

$$
\begin{equation*}
\delta_{y+m e_{1}} \geqslant^{\prime} \delta_{x} \geqslant^{\prime} \delta_{y-m e_{1}} . \tag{13}
\end{equation*}
$$

Proof. All three points follow from the definition of $\geqslant^{\prime}$ and M-NCI, Monotonicity, and Monetary equivalent, respectively.

## Aumann Utilities and Multi-Expected Utility Representations

In this section, in our formal results, we consider a binary relation $\geqslant^{*}$ over $\Delta$ such that

$$
\begin{equation*}
p \geqslant^{*} q \Longleftrightarrow \mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v) \quad \forall v \in \mathcal{W} \tag{14}
\end{equation*}
$$

where $\mathcal{W} \subseteq C\left(\mathbb{R}^{k}\right)$. Recall that a function $v \in C\left(\mathbb{R}^{k}\right)$ is an Aumann utility if and only if

$$
p>^{*} q \Longrightarrow \mathbb{E}_{p}(v)>\mathbb{E}_{q}(v) \text { and } p \sim^{*} q \Longrightarrow \mathbb{E}_{p}(v)=\mathbb{E}_{q}(v)
$$

We denote by $e$ the vector whose components are all 1 s . We endow $C\left(\mathbb{R}^{k}\right)$ with the distance $d: C\left(\mathbb{R}^{k}\right) \times C\left(\mathbb{R}^{k}\right) \rightarrow[0, \infty)$ defined by

$$
d(f, g)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n} \min \left\{\max _{x \in[-n e, n e]}|f(x)-g(x)|, 1\right\} \quad \forall f, g \in C\left(\mathbb{R}^{k}\right)
$$

It is routine to show that $\left(C\left(\mathbb{R}^{k}\right), d\right)$ is separable. ${ }^{24}$ Moreover, if $\left\{f_{m}\right\}_{m \in \mathbb{N}} \subseteq C\left(\mathbb{R}^{k}\right)$ is such that $f_{m} \xrightarrow{d} f$, then $\left\{f_{m}\right\}_{m \in \mathbb{N}}$ converges uniformly to $f$ on each compact subset of $\mathbb{R}^{k}$.

Proposition 7. If $\geqslant^{*}$ is as in (14) and such that

$$
\begin{equation*}
x>y \Longrightarrow \delta_{x}>^{*} \delta_{y} \tag{15}
\end{equation*}
$$

then $\geqslant^{*}$ admits a strictly increasing Aumann utility.
Proof. By (14), observe that $x>y$ implies $v(x) \geq v(y)$ for all $v \in \mathcal{W}$. This implies that each $v \in \mathcal{W}$ is increasing. By Aliprantis and Border (2006, Corollary 3.5), there exists a countable $d$-dense subset $D$ of $\mathcal{W}$. Clearly, we have that

$$
\begin{equation*}
p \geqslant^{*} q \Longrightarrow \mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v) \quad \forall v \in D . \tag{16}
\end{equation*}
$$

Vice-versa, consider $p, q \in \Delta$ such that $\mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v)$ for all $v \in D$. Since $p$ and $q$ have compact support, there exists $\bar{n} \in \mathbb{N}$ such that $[-\bar{n} e, \bar{n} e]$ contains both supports. Consider $v \in \mathcal{W}$. Since $D$ is $d$-dense in $\mathcal{W}$, there exists a sequence $\left\{v_{l}\right\}_{l \in \mathbb{N}} \subseteq D$ such that $v_{l} \xrightarrow{d} v$. It follows that $v_{l}$ converges uniformly on $[-\bar{n} e, \bar{n} e]$. This implies that

$$
\begin{aligned}
\mathbb{E}_{p}(v) & =\int_{[-\bar{n} e, \bar{n} e]} v \mathrm{~d} p=\lim _{l} \int_{[-\bar{n} e, \bar{n} e]} v_{l} \mathrm{~d} p=\lim _{l} \mathbb{E}_{p}\left(v_{l}\right) \\
& \geq \lim _{l} \mathbb{E}_{q}\left(v_{l}\right)=\lim _{l} \int_{[-\bar{n} e, \bar{n} e]} v_{l} \mathrm{~d} q=\int_{[-\bar{n} e, \bar{n} e]} v \mathrm{~d} q=\mathbb{E}_{q}(v) .
\end{aligned}
$$

[^0]By (14) and (16) and since $v$ was arbitrarily chosen, we can conclude that

$$
\begin{equation*}
p \geqslant^{*} q \Longleftrightarrow \mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v) \quad \forall v \in D . \tag{17}
\end{equation*}
$$

Since $D$ is countable, we can list its elements: $D=\left\{v_{m}\right\}_{m \in \mathbb{N}}$. Set $b_{l}=l+\max \left\{\left|v_{l}(-l e)\right|,\left|v_{l}(l e)\right|\right\}$ for all $l \in \mathbb{N}$ and $a_{m}=\Pi_{l=1}^{m} b_{l} \geq b_{m}$ for all $m \in \mathbb{N}$. Finally, define $v: \mathbb{R}^{k} \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
v(x)=\sum_{m=1}^{\infty} \frac{v_{m}(x)}{a_{m}} \quad \forall x \in \mathbb{R}^{k} \tag{18}
\end{equation*}
$$

We first prove that $v$ is a well-defined continuous function. Fix $x \in \mathbb{R}^{k}$. It follows that there exists $\bar{m} \in \mathbb{N}$ such that $x \in[-m e, m e]$ for all $m \geq \bar{m}$. Since each $v_{m}$ is increasing, we have that $\left|v_{m}(x)\right| \leq \max \left\{\left|v_{m}(-m e)\right|,\left|v_{m}(m e)\right|\right\} \leq b_{m} \leq a_{m}$ for all $m \geq \bar{m}$. Since $a_{m} \geq m$ ! for all $m \in \mathbb{N}$, it follows that

$$
\frac{\left|v_{m}(x)\right|}{a_{m}}=\frac{\left|v_{m}(x)\right|}{b_{m} a_{m-1}} \leq \frac{1}{a_{m-1}} \leq \frac{1}{(m-1)!} \quad \forall m \geq \bar{m}+1 .
$$

This implies that the right-hand side of (18) converges. Since $x$ was arbitrarily chosen, $v$ is well-defined. Next, consider $n \in \mathbb{N}$. From the same argument above, we have that

$$
\frac{\left|v_{m}(x)\right|}{a_{m}} \leq \frac{1}{(m-1)!} \quad \forall x \in[-n e, n e], \forall m \geq n+1
$$

By Weierstrass' $M$-test and since $\left\{v_{m} / a_{m}\right\}_{m \in \mathbb{N}}$ is a sequence of continuous functions, we can conclude that $v=\sum_{m=1}^{\infty} \frac{v_{m}}{a_{m}}$ converges uniformly on [ $-n e, n e$ ], yielding that $v$ is continuous on $[-n e, n e]$. Since $n$ was arbitrarily chosen, it follows that $v$ is continuous.

Finally, assume that $p>^{*} q$ (resp. $p \sim^{*} q$ ). By (17), we have that $\mathbb{E}_{p}\left(v_{m}\right) \geq \mathbb{E}_{q}\left(v_{m}\right)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_{p}\left(v_{\hat{m}}\right)>\mathbb{E}_{q}\left(v_{\hat{m}}\right)$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_{p}\left(v_{m}\right)=\mathbb{E}_{q}\left(v_{m}\right)$ for all $m \in \mathbb{N}$ ). In particular, we have that $\mathbb{E}_{p}\left(v_{m} / a_{m}\right) \geq \mathbb{E}_{q}\left(v_{m} / a_{m}\right)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_{p}\left(v_{\hat{m}} / a_{\hat{m}}\right)>$ $\mathbb{E}_{q}\left(v_{\hat{m}} / a_{\hat{m}}\right)$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_{p}\left(v_{m} / a_{m}\right)=\mathbb{E}_{q}\left(v_{m} / a_{m}\right)$ for all $m \in \mathbb{N}$ ). Since $\sum_{m=1}^{\infty} \frac{v_{m}}{a_{m}}$ converges uniformly on compacta and the supports of $p$ and $q$ are compact, we can conclude
that

$$
\begin{aligned}
\mathbb{E}_{p}(v)-\mathbb{E}_{q}(v) & =\mathbb{E}_{p}\left(\sum_{m=1}^{\infty} \frac{v_{m}}{a_{m}}\right)-\mathbb{E}_{q}\left(\sum_{m=1}^{\infty} \frac{v_{m}}{a_{m}}\right)=\lim _{l} \sum_{m=1}^{l} \mathbb{E}_{p}\left(\frac{v_{m}}{a_{m}}\right)-\lim _{l} \sum_{m=1}^{l} \mathbb{E}_{q}\left(\frac{v_{m}}{a_{m}}\right) \\
& =\lim _{l}\left[\sum_{m=1}^{l}\left(\mathbb{E}_{p}\left(\frac{v_{m}}{a_{m}}\right)-\mathbb{E}_{q}\left(\frac{v_{m}}{a_{m}}\right)\right)\right] .
\end{aligned}
$$

This implies that if $p>^{*} q$ (resp. $p \sim^{*} q$ ), then $\mathbb{E}_{p}(v)>\mathbb{E}_{q}(v)$ (resp. $\mathbb{E}_{p}(v)=\mathbb{E}_{q}(v)$ ), proving that $v$ is an Aumann utility. In particular, by (15), $v$ is strictly increasing.

Consider a binary relation $\geqslant^{*}$ on $\Delta$. Define $\mathcal{W}_{\text {max }}\left(\geqslant^{*}\right)$ as the set of all strictly increasing functions $v \in C\left(\mathbb{R}^{k}\right)$ such that $v(0)=0$ and $p \geqslant^{*} q$ implies $\mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v)$. We say that a set $\mathcal{W}$ in $C\left(\mathbb{R}^{k}\right)$ has full image if and only if

$$
\forall x, y \in \mathbb{R}^{k}, \exists m \in \mathbb{R}_{+} \text {s.t. } v\left(y+m e_{1}\right) \geq v(x) \geq v\left(y-m e_{1}\right) \quad \forall v \in \mathcal{W} .
$$

Proposition 8. Let $\geqslant^{*}$ be a binary relation on $\Delta$ represented as in (14). If $\geqslant^{*}$ satisfies (12) and (13), then $\mathcal{W}_{\max }\left(\geqslant^{*}\right)$ is a nonempty convex set with full image that satisfies (14).

Proof. Consider $v_{1}, v_{2} \in \mathcal{W}_{\max }\left(\geqslant^{*}\right)$ and $\lambda \in(0,1)$. Since both functions are strictly increasing and continuous and such that $v_{1}(0)=0=v_{2}(0)$, it follows that $\lambda v_{1}+(1-\lambda) v_{2}$ is strictly increasing, continuous, and takes value 0 in 0 . Since $v_{1}, v_{2} \in \mathcal{W}_{\max }\left(\geqslant^{*}\right)$, if $p \geqslant^{*} q$, then $\mathbb{E}_{p}\left(v_{1}\right) \geq \mathbb{E}_{q}\left(v_{1}\right)$ and $\mathbb{E}_{p}\left(v_{2}\right) \geq \mathbb{E}_{q}\left(v_{2}\right)$. This implies that

$$
\begin{aligned}
\mathbb{E}_{p}\left(\lambda v_{1}+(1-\lambda) v_{2}\right) & =\lambda \mathbb{E}_{p}\left(v_{1}\right)+(1-\lambda) \mathbb{E}_{p}\left(v_{2}\right) \\
& \geq \lambda \mathbb{E}_{q}\left(v_{1}\right)+(1-\lambda) \mathbb{E}_{q}\left(v_{2}\right)=\mathbb{E}_{q}\left(\lambda v_{1}+(1-\lambda) v_{2}\right),
\end{aligned}
$$

proving that $\lambda v_{1}+(1-\lambda) v_{2} \in \mathcal{W}_{\max }\left(\geqslant^{*}\right)$ and, in particular, $\mathcal{W}_{\max }\left(\geqslant^{*}\right)$ is convex. By Proposition 7, there exists a strictly increasing $\hat{v} \in C\left(\mathbb{R}^{k}\right)$ such that

$$
p>^{*} q \Longrightarrow \mathbb{E}_{p}(\hat{v})>\mathbb{E}_{q}(\hat{v}) \text { and } p \sim^{*} q \Longrightarrow \mathbb{E}_{p}(\hat{v})=\mathbb{E}_{q}(\hat{v}) .
$$

Without loss of generality, we can assume that $\hat{v}(0)=0$ (given $\hat{v}$, set $v=\hat{v}-\hat{v}(0)$ ) and, in particular, we have that $\hat{v} \in \mathcal{W}_{\text {max }}\left(\geqslant^{*}\right)$, proving that $\mathcal{W}_{\max }\left(\geqslant^{*}\right)$ is nonempty. Since $\geqslant^{*}$ satisfies (13), it follows that $\mathcal{W}_{\max }\left(\geqslant^{*}\right)$ has full image. Since $\geqslant^{*}$ satisfies (12), $v$ is increasing for all $v \in \mathcal{W}$. This implies that for each $v \in \mathcal{W}$ and for each $n \in \mathbb{N}$ the function
$v_{n}=\left(1-\frac{1}{n}\right) v+\frac{1}{n} \hat{v}-\left[\left(1-\frac{1}{n}\right) v(0)+\frac{1}{n} \hat{v}(0)\right] \in \mathcal{W}_{\text {max }}\left(\geqslant^{*}\right)$. By definition, if $p \geqslant^{*} q$, then $\mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v)$ for all $v \in \mathcal{W}_{\max }\left(\geqslant^{*}\right)$. Vice-versa, we have that

$$
\begin{aligned}
& \mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v) \quad \forall v \in \mathcal{W}_{\max }\left(\geqslant^{*}\right) \\
& \Longrightarrow \mathbb{E}_{p}\left(\left(1-\frac{1}{n}\right) v+\frac{1}{n} \hat{v}\right) \geq \mathbb{E}_{q}\left(\left(1-\frac{1}{n}\right) v+\frac{1}{n} \hat{v}\right) \quad \forall v \in \mathcal{W}, \forall n \in \mathbb{N} \\
& \Longrightarrow \mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v) \quad \forall v \in \mathcal{W} \Longrightarrow p \geqslant^{*} q,
\end{aligned}
$$

proving that (14) holds with $\mathcal{W}_{\max }\left(\geqslant^{*}\right)$ in place of $\mathcal{W}$.
We conclude by discussing Multi-Expected Utility representations which feature odd sets. To do this, we make two simple observations. First, recall the map $\sigma: \Delta \rightarrow \Delta$, which swaps gains with losses, defined by

$$
\sigma(p)(B)=p(-B) \text { for all Borel subsets of } \mathbb{R}^{k} \text { and for all } p \in \Delta
$$

It is immediate to see that $\sigma$ is affine and $\sigma(\sigma(p))=p$ for all $p \in \Delta$. Second, by the Change of Variable Theorem (see, e.g., Aliprantis and Border 2006, Theorem 13.46), we have that

$$
\begin{equation*}
\mathbb{E}_{\sigma(r)}(v)=\int_{\mathbb{R}^{k}} v \mathrm{~d} \sigma(r)=-\int_{\mathbb{R}^{k}} \bar{v} \mathrm{~d} r=-\mathbb{E}_{r}(\bar{v}) \quad \forall r \in \Delta, \forall v \in C\left(\mathbb{R}^{k}\right) \tag{19}
\end{equation*}
$$

where $\bar{v}: \mathbb{R}^{k} \rightarrow \mathbb{R}$ is defined by $\bar{v}(x)=-v(-x)$ for all $x \in \mathbb{R}^{k}$ and for all $v \in C\left(\mathbb{R}^{k}\right)$.
Proposition 9. Let $\geqslant^{*}$ be a binary relation on $\Delta$ represented as in (14) which satisfies (12) and (13). The following statements are equivalent:
(i) For each $p, q \in \Delta$

$$
p \geqslant^{*} q \Longleftrightarrow \sigma(q) \geqslant^{*} \sigma(p) .
$$

(ii) For each $p, q \in \Delta$

$$
p \geqslant^{*} q \Longrightarrow \sigma(q) \geqslant^{*} \sigma(p) .
$$

(iii) $\mathcal{W}_{\max }\left(\geqslant^{*}\right)$ is odd.

Moreover, if $\mathcal{W}$ in (14) is odd, then (i) and (ii) hold.
For the last part of the statement, that is proving that if $\mathcal{W}$ is odd, then (i) and (ii) hold, we can dispense with the assumption that $\geqslant^{*}$ satisfies (12) and (13). The proof will clarify.

Proof. By Proposition 8, we have that

$$
p \geqslant \geqslant^{*} q \Longleftrightarrow \mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v) \quad \forall v \in \mathcal{W}_{\max }\left(\geqslant^{*}\right) .
$$

In other words, for the first part of the statement, we can replace $\mathcal{W}$ in (14) with $\mathcal{W}_{\max }\left(\geqslant^{*}\right)$.
(i) implies (ii). It is obvious.
(ii) implies (iii). Fix $v \in \mathcal{W}_{\max }\left(\geqslant^{*}\right)$. By definition of $\bar{v}$ and since each $v$ in $\mathcal{W}_{\max }\left(\geqslant^{*}\right)$ is strictly increasing, continuous, and such that $v(0)=0$, we have that $\bar{v}$ is strictly increasing, continuous, and such that $\bar{v}(0)=0$. By assumption and (19), we have that
$p \geqslant^{*} q \Longrightarrow \sigma(q) \geqslant^{*} \sigma(p) \Longrightarrow \mathbb{E}_{\sigma(q)}(v) \geq \mathbb{E}_{\sigma(p)}(v) \Longrightarrow-\mathbb{E}_{q}(\bar{v}) \geq-\mathbb{E}_{p}(\bar{v}) \Longrightarrow \mathbb{E}_{p}(\bar{v}) \geq \mathbb{E}_{q}(\bar{v})$.
By definition of $\mathcal{W}_{\max }\left(\geqslant^{*}\right)$, we can conclude that $\bar{v} \in \mathcal{W}_{\max }\left(\geqslant^{*}\right)$, proving that $\mathcal{W}_{\max }\left(\geqslant^{*}\right)$ is odd.
(iii) implies (i). By (19) and since $\mathcal{W}$ is odd and represents $\geqslant^{*}$, we have that

$$
\begin{aligned}
p \geqslant^{*} q & \Longleftrightarrow \mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v) \quad \forall v \in \mathcal{W} \\
\Longleftrightarrow \mathbb{E}_{\sigma(q)}(v) \geq \mathbb{E}_{\sigma(p)}(v) \quad \forall v \in \mathcal{W} & \left.\Longleftrightarrow \sigma(q) \mathbb{E}_{p}(\bar{v}) \geq \mathbb{E}_{q}(\bar{v}) \quad \forall v \in \mathcal{W}\right),
\end{aligned}
$$

proving the implication (since $\mathcal{W}_{\max }\left(\geqslant^{*}\right)$ represents $\geqslant^{*}$ ) and also the second part of the statement.

## Representing $\geqslant^{\prime}$

We can finally provide a Multi-Expected Utility representation for $\geqslant^{\prime}$.
Proposition 10. If $\geqslant$ satisfies Weak Order, Continuity, Monotonicity, and Monetary equivalent, then

$$
p \geqslant^{\prime} q \Longleftrightarrow \mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v) \quad \forall v \in \mathcal{W}_{\max }\left(\geqslant^{\prime}\right) .
$$

Moreover, $\mathcal{W}_{\max }\left(\geqslant^{\prime}\right)$ is a nonempty convex set with full image.
Proof. By the same techniques of Cerreia-Vioglio (2009, Proposition 22) (see also CerreiaVioglio et al. 2017, Lemma 1 and Footnote 10), $\geqslant^{\prime}$ is a preorder that satisfies Sequential Continuity and Independence. ${ }^{25}$ By Evren (2008, Theorem 2), there exists a set $\mathcal{W} \subseteq$
${ }^{25}$ That is, for each two generalized sequences $\left\{p_{\alpha}\right\}_{\alpha \in A}$ and $\left\{q_{\alpha}\right\}_{\alpha \in A}$ in $\Delta$

$$
p_{\alpha} \geqslant^{\prime} q_{\alpha} \quad \forall \alpha \in A, p_{\alpha} \rightarrow p, \text { and } q_{\alpha} \rightarrow q \Longrightarrow p \geqslant^{\prime} q
$$

$C\left(\mathbb{R}^{k}\right)$ such that $p \geqslant^{\prime} q$ if and only if $\mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v)$ for all $v \in \mathcal{W}$. By Lemma 1 and since $\geqslant$ is a Weak Order which satisfies Monotonicity and Monetary equivalent, we have that $\geqslant^{\prime}$ satisfies (12) and (13). By Proposition 8 and considering $\geqslant^{\prime}$ in place of $\geqslant^{*}$, $\mathcal{W}$ can be chosen to be $\mathcal{W}_{\max }\left(\succcurlyeq^{\prime}\right)$, proving the statement.

## Missing Proofs

In this section, we prove Proposition 4. We begin by showing that if $\geqslant$ admits a finite essential Cautious Utility representation, then it is canonical. This fact will be key in proving the aforementioned proposition.

Lemma 2. If $\geqslant$ admits a finite essential Cautious Utility representation, then it is canonical.
Proof. Define $\geqslant^{*}$ to be such that $p \geqslant^{*} q$ if and only if $\mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v)$ for all $v \in \mathcal{W}$ where $\mathcal{W}$ is a finite essential Cautious Utility representation of $\geqslant$. Since $\mathcal{W}$ is finite, we have that the smallest convex cone containing $\mathcal{W}$, denoted by cone $(\mathcal{W})$, is closed with respect to the $\sigma\left(C\left(\mathbb{R}^{k}\right), \Delta\right)$-topology and so is the set cone $(\mathcal{W})+\left\{\theta 1_{\mathbb{R}^{k}}\right\}_{\theta \in \mathbb{R}}$. By definition of $\mathcal{W}_{\text {max }}\left(\geqslant^{*}\right)$, it follows that cone $(\mathcal{W}) \backslash\{0\} \subseteq \mathcal{W}_{\text {max }}\left(\geqslant^{*}\right)$. By Proposition 8, Remark 4, and (Evren, 2008, Theorem 5) and since $\mathcal{W}$ is a Cautious Utility representation, we have that (where the closure is in the $\sigma\left(C\left(\mathbb{R}^{k}\right), \Delta\right)$-topology)

$$
\begin{aligned}
\operatorname{cone}(\mathcal{W})+\left\{\theta 1_{\mathbb{R}^{k}}\right\}_{\theta \in \mathbb{R}} & =\operatorname{cl}\left(\operatorname{cone}\left(\mathcal{W}_{\max }\left(\geqslant^{*}\right)\right)+\left\{\theta 1_{\mathbb{R}^{k}}\right\}_{\theta \in \mathbb{R}}\right) \\
& \supseteq \operatorname{cl}\left(\mathcal{W}_{\max }\left(\geqslant^{*}\right)+\left\{\theta 1_{\mathbb{R}^{k}}\right\}_{\theta \in \mathbb{R}}\right) \\
& \supseteq \mathcal{W}_{\max }\left(\geqslant^{*}\right)+\left\{\theta 1_{\mathbb{R}^{k}}\right\}_{\theta \in \mathbb{R}},
\end{aligned}
$$

yielding that cone $(\mathcal{W}) \backslash\{0\} \supseteq \mathcal{W}_{\text {max }}\left(\geqslant^{*}\right)$ and, in particular, cone $(\mathcal{W}) \backslash\{0\}=\mathcal{W}_{\max }\left(\geqslant^{*}\right)$. Since the functional $v \mapsto c(p, v)$ is quasiconcave over cone $(\mathcal{W}) \backslash\{0\}$ for all $p \in \Delta$, it is immediate to see that

$$
V(p)=\min _{v \in \mathcal{W}} c(p, v)=\min _{v \in \operatorname{cone}(\mathcal{W}) \backslash\{0\}} c(p, v) \quad \forall p \in \Delta .
$$

By Remark 4 and since $\mathcal{W}=\left\{v_{i}\right\}_{i=1}^{n}$ is a finite Cautious Utility representation, we have that $\geqslant$ satisfies Axioms $1-5$. By Theorem 1 and its proof, $\mathcal{W}_{\max }\left(\geqslant^{\prime}\right)$ is a canonical Cautious

Utility representation for $\geqslant$. In particular, we have that

$$
V(p)=\min _{v \in \mathcal{W}} c(p, v)=\min _{v \in \operatorname{cone}(\mathcal{W}) \backslash\{0\}} c(p, v)=\inf _{v \in \mathcal{W}_{\max }\left(\forall^{\prime}\right)} c(p, v) \quad \forall p \in \Delta .
$$

Since $\geqslant^{\prime}$ is the largest subrelation of $\geqslant$ that satisfies the Independence axiom and $p \geqslant^{*} q$ implies $p \geqslant q$, we have that $\geqslant^{*}$ is a subrelation of $\geqslant^{\prime}$ and $\mathcal{W}_{\text {max }}\left(\succcurlyeq^{\prime}\right) \subseteq \mathcal{W}_{\text {max }}\left(\geqslant^{*}\right)=$ cone $(\mathcal{W}) \backslash\{0\}$. By contradiction, assume that $\mathcal{W}_{\text {max }}\left(\succcurlyeq^{\prime}\right) \neq$ cone $(\mathcal{W}) \backslash\{0\}$. Since $\mathcal{W}_{\max }\left(\succcurlyeq^{\prime}\right)$ is a convex set closed with respect to strictly positive scalar multiplications, this implies that $\mathcal{W} \nsubseteq \mathcal{W}_{\text {max }}\left(\geqslant^{\prime}\right)$. If $\mathcal{W}$ is a singleton, then $\geqslant$ is Expected Utility and, in particular, $\geqslant^{\prime}$ is complete and coincides with $\geqslant$. This implies that $\mathcal{W}=\left\{v_{1}\right\}$ and $\mathcal{W}_{\max }\left(\geqslant^{\prime}\right)=$ $\left\{\lambda \nu_{1}\right\}_{\lambda>0}=$ cone $(\mathcal{W}) \backslash\{0\}$, a contradiction. Assume $\mathcal{W}$ is not a singleton. Consider $\breve{v} \in \mathcal{W} \backslash \mathcal{W}_{\max }\left(\geqslant^{\prime}\right)$. Since $\mathcal{W}$ is essential, there exists $\bar{p} \in \Delta$ such that $\min _{v \in \mathcal{W}} c(\bar{p}, v)<$ $\min _{v \in \mathcal{W} \backslash\{\breve{v}\}} c(\bar{p}, v)$. Since $\mathcal{W}=\left\{v_{i}\right\}_{i=1}^{n}$ and $n \geq 2$, without loss of generality, we can set $\breve{v}=v_{n} \notin \mathcal{W}_{\text {max }}\left(\succcurlyeq^{\prime}\right)$. In particular, we have that

$$
\begin{equation*}
\inf _{v \in \mathcal{W}_{\max }\left(\geqslant^{\prime}\right)} c(\bar{p}, v)=\min _{v \in \mathcal{W}} c(\bar{p}, v)=c\left(\bar{p}, v_{n}\right)<c\left(\bar{p}, v_{i}\right) \quad \forall i \in\{1, \ldots, n-1\} . \tag{20}
\end{equation*}
$$

Consider a sequence $\left\{\hat{v}_{m}\right\}_{m \in \mathbb{N}} \subseteq \mathcal{W}_{\text {max }}\left(\geqslant^{\prime}\right)$ such that $c\left(\bar{p}, \hat{v}_{m}\right) \downarrow \inf _{v \in \mathcal{W}_{\text {max }}\left(\geqslant_{\prime}^{\prime}\right)} c(\bar{p}, v)$. By construction and since $\mathcal{W}_{\text {max }}\left(\succcurlyeq^{\prime}\right) \subseteq$ cone $(\mathcal{W}) \backslash\{0\}$, there exists a collection of scalars $\left\{\lambda_{m, i}\right\}_{m \in \mathbb{N}, i \in\{1, \ldots, n\}} \subseteq[0, \infty)$ such that $\hat{v}_{m}=\sum_{i=1}^{n} \lambda_{m, i} v_{i}$ for all $m \in \mathbb{N}$. Since $\hat{v}_{m}$ is strictly increasing, we have that for each $m \in \mathbb{N}$ there exists $i \in\{1, \ldots, n\}$ such that $\lambda_{m, i}>0$. Define $\lambda_{m, \sigma}=\sum_{i=1}^{n} \lambda_{m, i}>0$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ and for each $i \in\{1, \ldots, n\}$ define also $\bar{\lambda}_{m, i}=\lambda_{m, i} / \lambda_{m, \sigma}$ as well as $\tilde{v}_{m}=\sum_{i=1}^{n} \bar{\lambda}_{m, i} v_{i}=\hat{v}_{m} / \lambda_{m, \sigma}$. Since $\lambda_{m, \sigma}>0$ for all $m \in \mathbb{N}$, it is immediate to see that $c\left(\bar{p}, \tilde{v}_{m}\right)=c\left(\bar{p}, \hat{v}_{m}\right)$ for all $m \in \mathbb{N}$ and, in particular, $c\left(\bar{p}, \tilde{v}_{m}\right) \downarrow$ $\inf _{v \in \mathcal{W}_{\text {max }}\left(\geqslant_{\prime}\right) c} c(\bar{p}, v)$. For each $m \in \mathbb{N}$ denote by $\bar{\lambda}_{m}$ the $\mathbb{R}^{n}$ vector whose $i$-th component is $\bar{\lambda}_{m, i}$. Since $\left\{\bar{\lambda}_{m}\right\}_{m \in \mathbb{R}}$ is a sequence in the $\mathbb{R}^{n}$ simplex, there exists a subsequence $\left\{\bar{\lambda}_{m_{l}}\right\}_{l \in \mathbb{N}}$ such that $\bar{\lambda}_{m_{l}, i} \rightarrow \bar{\lambda}_{i} \in[0,1]$ for all $i \in\{1, \ldots, n\}$ and $\sum_{i=1}^{n} \bar{\lambda}_{i}=1$. It is immediate to see that $\tilde{v}_{m_{l}}=\sum_{i=1}^{n} \bar{\lambda}_{m_{l}, i} v_{i} \xrightarrow{\sigma\left(c\left(\mathbb{R}^{k}\right), \Delta\right)} \sum_{i=1}^{n} \bar{\lambda}_{i} v_{i}=\tilde{v}$ where $\tilde{v}$ is continuous, strictly increasing, and such that $\tilde{v}(0)=0$. Moreover, for each $p, q \in \Delta$ we have that $p \geqslant^{\prime} q$ implies $\mathbb{E}_{p}(\tilde{v}) \geq \mathbb{E}_{q}(\tilde{v})$, proving that $\tilde{v} \in \mathcal{W}_{\max }\left(\geqslant^{\prime}\right)$. Note that $\bar{\lambda}_{n}<1$, otherwise, we would have that $v_{n}=\tilde{v} \in$ $\mathcal{W}_{\text {max }}\left(\succcurlyeq^{\prime}\right)$, a contradiction. By (20) and since $\bar{\lambda}_{n}<1$ and the functional $v \mapsto c(p, v)$ is
explicitly quasiconcave over co $(\mathcal{W})$ for all $p \in \Delta,{ }^{26}$ we have that

$$
c\left(\bar{p}, v_{n}\right)<c(\bar{p}, \tilde{v})=\lim _{l} c\left(\bar{p}, \tilde{v}_{m_{l}}\right)=\lim _{m} c\left(\bar{p}, \tilde{v}_{m}\right)=\inf _{v \in \mathcal{W}_{\max }\left(\geqslant^{\prime}\right)} c(\bar{p}, v)=c\left(\bar{p}, v_{n}\right),
$$

a contradiction. It follows that $\mathcal{W}_{\text {max }}\left(\geqslant^{\prime}\right)=$ cone $(\mathcal{W}) \backslash\{0\}$ and, in particular, $\mathcal{W}$ represents also $\geqslant^{\prime}$. This implies that $\mathcal{W}$ is canonical.

Proof of Proposition 4. We first prove the first part of the statement assuming $\geqslant$ satisfies $\mathrm{u}-$ CPT, and then we will move to the additive case. Since $u(0)=0$ and $u$ is strictly increasing, it follows that there exists $\bar{t}>0$ such that $[-\bar{t}, \bar{t}] \subseteq \operatorname{Im} u$. Let $\Delta_{0}([0, \bar{t}])$ be the set of finitely supported probabilities over $[0, \bar{t}]$. Consider $\tilde{p} \in \Delta_{0}([0, \bar{t}])$. By definition, we have that there exist two unique collections $\left\{t_{i}\right\}_{i=1}^{n} \subseteq[0, \bar{t}]$ and $\left\{\lambda_{i}\right\}_{i=1}^{n} \subseteq[0,1]$ such that supp $p=$ $\left\{t_{i}\right\}_{i=1}^{n}, \sum_{i=1}^{n} \lambda_{i}=1$, and $\tilde{p}=\sum_{i=1}^{n} \lambda_{i} \delta_{t_{i}}$. Without loss of generality, we can assume that $t_{1}<\ldots<t_{n}$. We define $\tilde{V}: \Delta_{0}([0, \bar{t}]) \rightarrow \mathbb{R}$ by

$$
\tilde{V}(\tilde{p})=\sum_{j=1}^{n-1}\left(w^{+}\left(\sum_{i=j}^{n} \lambda_{i}\right)-w^{+}\left(\sum_{i=j+1}^{n} \lambda_{i}\right)\right) v\left(t_{j}\right)+w^{+}\left(\lambda_{n}\right) v\left(t_{n}\right)
$$

for all $\tilde{p} \in \Delta_{0}([0, \bar{t}])$. We next show that for each $\tilde{p} \in \Delta_{0}([0, \bar{t}])$ and for each $\tilde{t} \in[0, \bar{t}]$, if $\tilde{V}(\tilde{p})=\tilde{V}\left(\delta_{\tilde{t}}\right)$, then $\tilde{V}\left(\lambda \tilde{p}+(1-\lambda) \delta_{\tilde{t}}\right)=\tilde{V}\left(\delta_{\tilde{t}}\right)$ for all $\lambda \in(0,1)$. Consider $\tilde{p} \in \Delta_{0}([0, \bar{t}])$ and $\tilde{t} \in[0, \bar{t}]$ such that $\tilde{V}(\tilde{p})=\tilde{V}\left(\delta_{\tilde{t}}\right)$. Given $\tilde{p} \in \Delta_{0}([0, \bar{t}])$, since $\left\{t_{i}\right\}_{i=1}^{n} \subseteq[0, \bar{t}] \subseteq \operatorname{Im} u$, there exists $\left\{x_{i}\right\}_{i=1}^{n} \subseteq \mathbb{R}^{k}$ such that $u\left(x_{i}\right)=t_{i}$ for all $i \in\{1, \ldots, n\}$. Consider $p=\sum_{i=1}^{n} \lambda_{i} \delta_{x_{i}}$. It is immediate to see that $\tilde{V}(\tilde{p})=V(p)$. Since $\geqslant$ admits a Symmetric Cautious Utility representation, there exists $c \in \mathbb{R}$ such that $p \sim \delta_{c e_{1}}$. This implies that $V(p)=V\left(\delta_{c e_{1}}\right)$ and, in particular, $u\left(c e_{1}\right) \in[0, \bar{t}]$. Moreover, since $u$ and $v$ are strictly increasing, we have that $u\left(c e_{1}\right)=\tilde{t} \in[0, \bar{t}]$ and $V\left(\delta_{c e_{1}}\right)=\tilde{V}\left(\delta_{\tilde{t}}\right)$. By Remark 4 and since $\geqslant$ admits a Symmetric Cautious Utility representation, we have that $\geqslant$ satisfies M-NCI. This yields that $\lambda p+(1-\lambda) \delta_{c e_{1}} \sim \delta_{c e_{1}}$ for all $\lambda \in(0,1)$. This implies that

$$
\tilde{V}\left(\lambda \tilde{p}+(1-\lambda) \delta_{\tilde{t}}\right)=V\left(\lambda p+(1-\lambda) \delta_{c e_{1}}\right)=V\left(\delta_{c e_{1}}\right)=\tilde{V}\left(\delta_{\tilde{t}}\right) .
$$

${ }^{26}$ Formally, see e.g. (Aliprantis and Border, 2006, p. 300), given $p \in \Delta$, for each $h \in \mathbb{N} \backslash\{1\}$, for each $\left\{v_{l}\right\}_{l=1}^{h} \subseteq \operatorname{co}(\mathcal{W})$, and for each $\left\{\lambda_{l}\right\}_{l=1}^{h} \subseteq[0,1]$ such that $\sum_{l=1}^{h} \lambda_{l}=1$ and $\lambda_{h}<1$

$$
c\left(p, v_{i}\right)>c\left(p, v_{h}\right) \quad \forall i \in\{1, \ldots, h-1\} \Longrightarrow c\left(p, \sum_{i=1}^{h} \lambda_{i} v_{i}\right)>c\left(p, v_{h}\right) .
$$

By Bell and Fishburn (2003, Theorem 1) applied to $\tilde{V}$, it follows that $w^{+}$is the identity. The same proof, performed with $[-\bar{t}, 0]$ in place of $[0, \bar{t}]$ and $w^{+}$replaced by $w^{-}$, yields that $w^{-}$is the identity. These two facts together allow us to conclude that $p \mapsto V(p)=\mathrm{CPT}_{v, w^{+}, w^{-}}\left(p_{u}\right)$ is an Expected Utility functional with utility $v \circ u: \mathbb{R}^{k} \rightarrow \mathbb{R}$. We next assume that $\geqslant$ admits an Additive CPT representation. As before consider $\bar{t}>0$. Define $\Delta_{0}([0, \bar{t}])$ and $\tilde{V}$ as before with $v$ replaced by $u_{1}$. For each $\tilde{p} \in \Delta_{0}([0, \bar{t}])$ define $p$ in $\Delta$ to be the product measure $\tilde{p} \otimes \delta_{0} \ldots \otimes \delta_{0}$. It is immediate to see that $\tilde{V}(\tilde{p})=V(p)$ for all $\tilde{p} \in \Delta_{0}([0, \tilde{t}])$. As before, we can show that for each $\tilde{p} \in \Delta_{0}([0, \bar{t}])$ and for each $\tilde{t} \in[0, \bar{t}]$, if $\tilde{V}(\tilde{p})=\tilde{V}\left(\delta_{\tilde{t}}\right)$, then $\tilde{V}\left(\lambda \tilde{p}+(1-\lambda) \delta_{\tilde{t}}\right)=\tilde{V}\left(\delta_{\tilde{t}}\right)$ for all $\lambda \in(0,1)$. Consider $\tilde{p} \in \Delta_{0}([0, \tilde{t}])$ and $\tilde{t} \in[0, \tilde{t}]$ such that $\tilde{V}(\tilde{p})=\tilde{V}\left(\delta_{\tilde{t}}\right)$. This implies that $V(p)=V\left(\delta_{\tilde{t} e_{1}}\right)$, that is, $p \sim \delta_{\tilde{t} e_{1}}$. By Remark 4 and since $\geqslant$ admits a Symmetric Cautious Utility representation, we have that $\geqslant$ satisfies M-NCI. This yields that $\lambda p+(1-\lambda) \delta_{\tilde{t} e_{1}} \sim \delta_{\tilde{t} e_{1}}$ for all $\lambda \in(0,1)$. This implies that

$$
\tilde{V}\left(\lambda \tilde{p}+(1-\lambda) \delta_{\tilde{t}}\right)=V\left(\lambda p+(1-\lambda) \delta_{\tilde{t} e_{1}}\right)=V\left(\delta_{\tilde{t} e_{1}}\right)=\tilde{V}\left(\delta_{\tilde{t}}\right) .
$$

By Bell and Fishburn (2003, Theorem 1) applied to $\tilde{V}$, it follows that $w^{+}$is the identity. The same proof, performed with $[-\bar{t}, 0]$ in place of $[0, \bar{t}]$ and $w^{+}$replaced by $w^{-}$, yields that $w^{-}$ is the identity. This implies that $\geqslant$ admits an Expected Utility representation with utility $u: \mathbb{R}^{k} \rightarrow \mathbb{R}$ defined by $u(x)=\sum_{i=1}^{k} u_{i}\left(x_{i}\right)$ for all $x \in \mathbb{R}^{k}$.

As for the second part of the statement, by Lemma 2 and since $\mathcal{W}$ is a finite essential Cautious Utility representation, we have that $\mathcal{W}$ is a canonical representation, that is, $\mathcal{W}=$ $\left\{v_{i}\right\}_{i=1}^{n}$ represents also $\geqslant^{\prime}$. Since $\geqslant$ is Expected Utility with utility $v \circ u$ (where in the linear case $v$ is the identity and $u$ is additively separable), we have that $\geqslant^{\prime}$ coincides with $\geqslant$, yielding that for each $i \in\{1, \ldots, n\}$ there exists $\lambda_{i}>0$ such that $v_{i}=\lambda_{i}(v \circ u)$. This implies that $c\left(p, v_{i}\right)=c(p, v \circ u)$ for all $p \in \Delta$ and for all $i \in\{1, \ldots, n\}$. Since $\mathcal{W}$ is essential, this implies that $\mathcal{W}$ is a singleton. Since $\mathcal{W}=\left\{v_{1}\right\}$ and $\mathcal{W}$ is odd, this implies that $v_{1}$ is odd and, in particular, $\geqslant$ is loss neutral for risk and exhibits no endowment effect.

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[^0]:    ${ }^{24} \mathrm{~A}$ proof is available upon request.

