Online Appendix

This appendix includes all the missing proofs and the ancillary facts used in the main body of the paper. We begin with a section on facts instrumental for Theorem 1 and Proposition 5.

Foundation

Recall the definition of \geq' in Section 5, that is,

$$p \geq' q \iff \lambda p + (1 - \lambda) r \geq \lambda q + (1 - \lambda) r \qquad \forall \lambda \in (0, 1], \forall r \in \Delta.$$

The goal of this section is to provide a Multi-Expected Utility representation for \geq' .

Lemma 1. Let \geq be a binary relation on Δ that satisfies Weak Order. The following statements are true:

1. The relation \geq satisfies M-NCI if and only if for each $p \in \Delta$ and for each $m \in \mathbb{R}$

$$p \ge \delta_{me_1} \implies p \ge' \delta_{me_1}.$$
 (Equivalently $p \ge' \delta_{me_1} \implies \delta_{me_1} > p.$)

2. If \geq satisfies Monotonicity, then for each $x, y \in \mathbb{R}^k$

$$x > y \implies \delta_x >' \delta_y. \tag{12}$$

3. If \geq satisfies Monetary equivalent, then for each $x, y \in \mathbb{R}^k$ there exists $m \in \mathbb{R}_+$ such that

$$\delta_{y+me_1} \ge' \delta_x \ge' \delta_{y-me_1}. \tag{13}$$

Proof. All three points follow from the definition of \geq' and M-NCI, Monotonicity, and Monetary equivalent, respectively.

Aumann Utilities and Multi-Expected Utility Representations

In this section, in our formal results, we consider a binary relation \geq^* over Δ such that

$$p \geq^{*} q \iff \mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v) \quad \forall v \in \mathcal{W}$$
 (14)

where $\mathcal{W} \subseteq C(\mathbb{R}^k)$. Recall that a function $v \in C(\mathbb{R}^k)$ is an Aumann utility if and only if

$$p \succ^{*} q \implies \mathbb{E}_{p}(v) > \mathbb{E}_{q}(v) \text{ and } p \sim^{*} q \implies \mathbb{E}_{p}(v) = \mathbb{E}_{q}(v).$$

We denote by *e* the vector whose components are all 1s. We endow $C(\mathbb{R}^k)$ with the distance $d: C(\mathbb{R}^k) \times C(\mathbb{R}^k) \to [0, \infty)$ defined by

$$d\left(f,g\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n} \min\left\{\max_{x \in [-ne,ne]} \left|f\left(x\right) - g\left(x\right)\right|, 1\right\} \quad \forall f,g \in C\left(\mathbb{R}^{k}\right).$$

It is routine to show that $(C(\mathbb{R}^k), d)$ is separable.²⁴ Moreover, if $\{f_m\}_{m \in \mathbb{N}} \subseteq C(\mathbb{R}^k)$ is such that $f_m \xrightarrow{d} f$, then $\{f_m\}_{m \in \mathbb{N}}$ converges uniformly to f on each compact subset of \mathbb{R}^k .

Proposition 7. *If* \geq ^{*} *is as in (14) and such that*

$$x > y \implies \delta_x >^* \delta_y, \tag{15}$$

then \geq^* admits a strictly increasing Aumann utility.

Proof. By (14), observe that x > y implies $v(x) \ge v(y)$ for all $v \in W$. This implies that each $v \in W$ is increasing. By Aliprantis and Border (2006, Corollary 3.5), there exists a countable *d*-dense subset *D* of W. Clearly, we have that

$$p \geq^* q \implies \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in D.$$
 (16)

Vice-versa, consider $p, q \in \Delta$ such that $\mathbb{E}_p(v) \ge \mathbb{E}_q(v)$ for all $v \in D$. Since p and q have compact support, there exists $\bar{n} \in \mathbb{N}$ such that $[-\bar{n}e, \bar{n}e]$ contains both supports. Consider $v \in \mathcal{W}$. Since D is d-dense in \mathcal{W} , there exists a sequence $\{v_l\}_{l \in \mathbb{N}} \subseteq D$ such that $v_l \xrightarrow{d} v$. It follows that v_l converges uniformly on $[-\bar{n}e, \bar{n}e]$. This implies that

$$\mathbb{E}_{p}(v) = \int_{[-\bar{n}e,\bar{n}e]} v dp = \lim_{l} \int_{[-\bar{n}e,\bar{n}e]} v_{l} dp = \lim_{l} \mathbb{E}_{p}(v_{l})$$
$$\geq \lim_{l} \mathbb{E}_{q}(v_{l}) = \lim_{l} \int_{[-\bar{n}e,\bar{n}e]} v_{l} dq = \int_{[-\bar{n}e,\bar{n}e]} v dq = \mathbb{E}_{q}(v).$$

²⁴A proof is available upon request.

By (14) and (16) and since v was arbitrarily chosen, we can conclude that

$$p \geq^{*} q \iff \mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v) \quad \forall v \in D.$$
 (17)

Since *D* is countable, we can list its elements: $D = \{v_m\}_{m \in \mathbb{N}}$. Set $b_l = l + \max\{|v_l(-le)|, |v_l(le)|\}$ for all $l \in \mathbb{N}$ and $a_m = \prod_{l=1}^m b_l \ge b_m$ for all $m \in \mathbb{N}$. Finally, define $v : \mathbb{R}^k \to \mathbb{R}$ by

$$v(x) = \sum_{m=1}^{\infty} \frac{v_m(x)}{a_m} \qquad \forall x \in \mathbb{R}^k.$$
 (18)

We first prove that v is a well-defined continuous function. Fix $x \in \mathbb{R}^k$. It follows that there exists $\overline{m} \in \mathbb{N}$ such that $x \in [-me, me]$ for all $m \ge \overline{m}$. Since each v_m is increasing, we have that $|v_m(x)| \le \max \{|v_m(-me)|, |v_m(me)|\} \le b_m \le a_m$ for all $m \ge \overline{m}$. Since $a_m \ge m!$ for all $m \in \mathbb{N}$, it follows that

$$\frac{|v_m(x)|}{a_m} = \frac{|v_m(x)|}{b_m a_{m-1}} \le \frac{1}{a_{m-1}} \le \frac{1}{(m-1)!} \qquad \forall m \ge \bar{m} + 1.$$

This implies that the right-hand side of (18) converges. Since x was arbitrarily chosen, v is well-defined. Next, consider $n \in \mathbb{N}$. From the same argument above, we have that

$$\frac{\left|v_{m}\left(x\right)\right|}{a_{m}} \leq \frac{1}{(m-1)!} \quad \forall x \in \left[-ne, ne\right], \forall m \geq n+1.$$

By Weierstrass' *M*-test and since $\{v_m/a_m\}_{m\in\mathbb{N}}$ is a sequence of continuous functions, we can conclude that $v = \sum_{m=1}^{\infty} \frac{v_m}{a_m}$ converges uniformly on [-ne, ne], yielding that v is continuous on [-ne, ne]. Since n was arbitrarily chosen, it follows that v is continuous.

Finally, assume that $p >^* q$ (resp. $p \sim^* q$). By (17), we have that $\mathbb{E}_p(v_m) \ge \mathbb{E}_q(v_m)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_p(v_{\hat{m}}) > \mathbb{E}_q(v_{\hat{m}})$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_p(v_m) = \mathbb{E}_q(v_m)$ for all $m \in \mathbb{N}$). In particular, we have that $\mathbb{E}_p(v_m/a_m) \ge \mathbb{E}_q(v_m/a_m)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_p(v_{\hat{m}}/a_{\hat{m}}) > \mathbb{E}_q(v_{\hat{m}}/a_{\hat{m}})$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_p(v_m/a_m) = \mathbb{E}_q(v_m/a_m)$ for all $m \in \mathbb{N}$). Since $\sum_{m=1}^{\infty} \frac{v_m}{a_m}$ converges uniformly on compacta and the supports of p and q are compact, we can conclude that

$$\mathbb{E}_{p}(v) - \mathbb{E}_{q}(v) = \mathbb{E}_{p}\left(\sum_{m=1}^{\infty} \frac{v_{m}}{a_{m}}\right) - \mathbb{E}_{q}\left(\sum_{m=1}^{\infty} \frac{v_{m}}{a_{m}}\right) = \lim_{l} \sum_{m=1}^{l} \mathbb{E}_{p}\left(\frac{v_{m}}{a_{m}}\right) - \lim_{l} \sum_{m=1}^{l} \mathbb{E}_{q}\left(\frac{v_{m}}{a_{m}}\right)$$
$$= \lim_{l} \left[\sum_{m=1}^{l} \left(\mathbb{E}_{p}\left(\frac{v_{m}}{a_{m}}\right) - \mathbb{E}_{q}\left(\frac{v_{m}}{a_{m}}\right)\right)\right].$$

This implies that if $p >^* q$ (resp. $p \sim^* q$), then $\mathbb{E}_p(v) > \mathbb{E}_q(v)$ (resp. $\mathbb{E}_p(v) = \mathbb{E}_q(v)$), proving that v is an Aumann utility. In particular, by (15), v is strictly increasing.

Consider a binary relation \geq^* on Δ . Define \mathcal{W}_{\max} (\geq^*) as the set of all strictly increasing functions $v \in C(\mathbb{R}^k)$ such that v(0) = 0 and $p \geq^* q$ implies $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$. We say that a set \mathcal{W} in $C(\mathbb{R}^k)$ has full image if and only if

$$\forall x, y \in \mathbb{R}^k, \exists m \in \mathbb{R}_+ \text{ s.t. } v (y + me_1) \ge v (x) \ge v (y - me_1) \qquad \forall v \in \mathcal{W}.$$

Proposition 8. Let \geq^* be a binary relation on Δ represented as in (14). If \geq^* satisfies (12) and (13), then $W_{\max}(\geq^*)$ is a nonempty convex set with full image that satisfies (14).

Proof. Consider $v_1, v_2 \in W_{\max} (\geq^*)$ and $\lambda \in (0, 1)$. Since both functions are strictly increasing and continuous and such that $v_1(0) = 0 = v_2(0)$, it follows that $\lambda v_1 + (1 - \lambda) v_2$ is strictly increasing, continuous, and takes value 0 in 0. Since $v_1, v_2 \in W_{\max} (\geq^*)$, if $p \geq^* q$, then $\mathbb{E}_p(v_1) \geq \mathbb{E}_q(v_1)$ and $\mathbb{E}_p(v_2) \geq \mathbb{E}_q(v_2)$. This implies that

$$\mathbb{E}_{p} (\lambda v_{1} + (1 - \lambda) v_{2}) = \lambda \mathbb{E}_{p} (v_{1}) + (1 - \lambda) \mathbb{E}_{p} (v_{2})$$

$$\geq \lambda \mathbb{E}_{q} (v_{1}) + (1 - \lambda) \mathbb{E}_{q} (v_{2}) = \mathbb{E}_{q} (\lambda v_{1} + (1 - \lambda) v_{2}),$$

proving that $\lambda v_1 + (1 - \lambda) v_2 \in \mathcal{W}_{\max}(\geq^*)$ and, in particular, $\mathcal{W}_{\max}(\geq^*)$ is convex. By Proposition 7, there exists a strictly increasing $\hat{v} \in C(\mathbb{R}^k)$ such that

$$p >^{*} q \Longrightarrow \mathbb{E}_{p}(\hat{v}) > \mathbb{E}_{q}(\hat{v}) \text{ and } p \sim^{*} q \Longrightarrow \mathbb{E}_{p}(\hat{v}) = \mathbb{E}_{q}(\hat{v}).$$

Without loss of generality, we can assume that $\hat{v}(0) = 0$ (given \hat{v} , set $v = \hat{v} - \hat{v}(0)$) and, in particular, we have that $\hat{v} \in W_{\max}(\geq^*)$, proving that $W_{\max}(\geq^*)$ is nonempty. Since \geq^* satisfies (13), it follows that $W_{\max}(\geq^*)$ has full image. Since \geq^* satisfies (12), v is increasing for all $v \in W$. This implies that for each $v \in W$ and for each $n \in \mathbb{N}$ the function $v_n = \left(1 - \frac{1}{n}\right)v + \frac{1}{n}\hat{v} - \left[\left(1 - \frac{1}{n}\right)v\left(0\right) + \frac{1}{n}\hat{v}\left(0\right)\right] \in \mathcal{W}_{\max}\left(\geq^*\right).$ By definition, if $p \geq^* q$, then $\mathbb{E}_p\left(v\right) \geq \mathbb{E}_q\left(v\right)$ for all $v \in \mathcal{W}_{\max}\left(\geq^*\right).$ Vice-versa, we have that

$$\begin{split} \mathbb{E}_{p}\left(v\right) \geq \mathbb{E}_{q}\left(v\right) & \forall v \in \mathcal{W}_{\max}\left(\geq^{*}\right) \\ \implies \mathbb{E}_{p}\left(\left(1-\frac{1}{n}\right)v + \frac{1}{n}\hat{v}\right) \geq \mathbb{E}_{q}\left(\left(1-\frac{1}{n}\right)v + \frac{1}{n}\hat{v}\right) & \forall v \in \mathcal{W}, \forall n \in \mathbb{N} \\ \implies \mathbb{E}_{p}\left(v\right) \geq \mathbb{E}_{q}\left(v\right) & \forall v \in \mathcal{W} \implies p \geq^{*} q, \end{split}$$

proving that (14) holds with $\mathcal{W}_{max} (\geq^*)$ in place of \mathcal{W} .

We conclude by discussing Multi-Expected Utility representations which feature odd sets. To do this, we make two simple observations. First, recall the map $\sigma : \Delta \to \Delta$, which swaps gains with losses, defined by

 $\sigma(p)(B) = p(-B)$ for all Borel subsets of \mathbb{R}^k and for all $p \in \Delta$.

It is immediate to see that σ is affine and $\sigma(\sigma(p)) = p$ for all $p \in \Delta$. Second, by the Change of Variable Theorem (see, e.g., Aliprantis and Border 2006, Theorem 13.46), we have that

$$\mathbb{E}_{\sigma(r)}(v) = \int_{\mathbb{R}^{k}} v \mathrm{d}\sigma(r) = -\int_{\mathbb{R}^{k}} \bar{v} \mathrm{d}r = -\mathbb{E}_{r}(\bar{v}) \quad \forall r \in \Delta, \forall v \in C\left(\mathbb{R}^{k}\right)$$
(19)

where $\bar{v} : \mathbb{R}^k \to \mathbb{R}$ is defined by $\bar{v}(x) = -v(-x)$ for all $x \in \mathbb{R}^k$ and for all $v \in C(\mathbb{R}^k)$.

Proposition 9. Let \geq^* be a binary relation on Δ represented as in (14) which satisfies (12) and (13). The following statements are equivalent:

(i) For each $p, q \in \Delta$

$$p \geqslant^{*} q \iff \sigma\left(q\right) \geqslant^{*} \sigma\left(p\right).$$

(ii) For each $p, q \in \Delta$

$$p \geq ^{*} q \implies \sigma(q) \geq ^{*} \sigma(p)$$
.

(iii) $W_{\max}(\geq^*)$ is odd.

Moreover, if W in (14) is odd, then (i) and (ii) hold.

For the last part of the statement, that is proving that if \mathcal{W} is odd, then (i) and (ii) hold, we can dispense with the assumption that \geq^* satisfies (12) and (13). The proof will clarify.

Proof. By Proposition 8, we have that

$$p \geq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\geq^*).$$

In other words, for the first part of the statement, we can replace \mathcal{W} in (14) with $\mathcal{W}_{max} (\geq^*)$.

(i) implies (ii). It is obvious.

(ii) implies (iii). Fix $v \in W_{\max}(\geq^*)$. By definition of \bar{v} and since each v in $W_{\max}(\geq^*)$ is strictly increasing, continuous, and such that v(0) = 0, we have that \bar{v} is strictly increasing, continuous, and such that $\bar{v}(0) = 0$. By assumption and (19), we have that

$$p \geq^{*} q \implies \sigma\left(q\right) \geq^{*} \sigma\left(p\right) \implies \mathbb{E}_{\sigma\left(q\right)}\left(v\right) \geq \mathbb{E}_{\sigma\left(p\right)}\left(v\right) \implies -\mathbb{E}_{q}\left(\bar{v}\right) \geq -\mathbb{E}_{p}\left(\bar{v}\right) \implies \mathbb{E}_{p}\left(\bar{v}\right) \geq \mathbb{E}_{q}\left(\bar{v}\right) = 0$$

By definition of $\mathcal{W}_{\max}(\geq^*)$, we can conclude that $\bar{v} \in \mathcal{W}_{\max}(\geq^*)$, proving that $\mathcal{W}_{\max}(\geq^*)$ is odd.

(iii) implies (i). By (19) and since \mathcal{W} is odd and represents \geq^* , we have that

$$p \geq^{*} q \iff \mathbb{E}_{p}(v) \geq \mathbb{E}_{q}(v) \quad \forall v \in \mathcal{W} \iff \mathbb{E}_{p}(\bar{v}) \geq \mathbb{E}_{q}(\bar{v}) \quad \forall v \in \mathcal{W}$$
$$\iff \mathbb{E}_{\sigma(q)}(v) \geq \mathbb{E}_{\sigma(p)}(v) \quad \forall v \in \mathcal{W} \iff \sigma(q) \geq^{*} \sigma(p),$$

proving the implication (since $\mathcal{W}_{max} (\geq^*)$ represents \geq^*) and also the second part of the statement.

Representing \geq'

We can finally provide a Multi-Expected Utility representation for \geq' .

Proposition 10. If \geq satisfies Weak Order, Continuity, Monotonicity, and Monetary equivalent, then

 $p \geq ' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\geq').$

Moreover, $\mathcal{W}_{\max}(\geq')$ is a nonempty convex set with full image.

Proof. By the same techniques of Cerreia-Vioglio (2009, Proposition 22) (see also Cerreia-Vioglio et al. 2017, Lemma 1 and Footnote 10), \geq' is a preorder that satisfies Sequential Continuity and Independence.²⁵ By Evren (2008, Theorem 2), there exists a set $\mathcal{W} \subseteq$

$$p_{\alpha} \geq ' q_{\alpha} \quad \forall \alpha \in A, \, p_{\alpha} \to p, \, \text{and} \, q_{\alpha} \to q \implies p \geq ' q.$$

 $^{^{25}}$ That is, for each two generalized sequences $\{p_{\alpha}\}_{\alpha\in A}$ and $\{q_{\alpha}\}_{\alpha\in A}$ in Δ

 $C(\mathbb{R}^k)$ such that $p \ge 'q$ if and only if $\mathbb{E}_p(v) \ge \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$. By Lemma 1 and since \ge is a Weak Order which satisfies Monotonicity and Monetary equivalent, we have that \ge' satisfies (12) and (13). By Proposition 8 and considering \ge' in place of \ge^* , \mathcal{W} can be chosen to be $\mathcal{W}_{\max}(\ge')$, proving the statement.

Missing Proofs

In this section, we prove Proposition 4. We begin by showing that if \geq admits a finite essential Cautious Utility representation, then it is canonical. This fact will be key in proving the aforementioned proposition.

Lemma 2. If \geq admits a finite essential Cautious Utility representation, then it is canonical.

Proof. Define \geq^* to be such that $p \geq^* q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in W$ where W is a finite essential Cautious Utility representation of \geq . Since W is finite, we have that the smallest convex cone containing W, denoted by cone (W), is closed with respect to the $\sigma(C(\mathbb{R}^k), \Delta)$ -topology and so is the set cone $(W) + \{\theta \mathbb{1}_{\mathbb{R}^k}\}_{\theta \in \mathbb{R}}$. By definition of $W_{\max}(\geq^*)$, it follows that cone $(W) \setminus \{0\} \subseteq W_{\max}(\geq^*)$. By Proposition 8, Remark 4, and (Evren, 2008, Theorem 5) and since W is a Cautious Utility representation, we have that (where the closure is in the $\sigma(C(\mathbb{R}^k), \Delta)$ -topology)

$$\operatorname{cone} \left(\mathcal{W} \right) + \left\{ \theta \mathbb{1}_{\mathbb{R}^{k}} \right\}_{\theta \in \mathbb{R}} = \operatorname{cl} \left(\operatorname{cone} \left(\mathcal{W}_{\max} \left(\geq^{*} \right) \right) + \left\{ \theta \mathbb{1}_{\mathbb{R}^{k}} \right\}_{\theta \in \mathbb{R}} \right)$$
$$\supseteq \operatorname{cl} \left(\mathcal{W}_{\max} \left(\geq^{*} \right) + \left\{ \theta \mathbb{1}_{\mathbb{R}^{k}} \right\}_{\theta \in \mathbb{R}} \right)$$
$$\supseteq \mathcal{W}_{\max} \left(\geq^{*} \right) + \left\{ \theta \mathbb{1}_{\mathbb{R}^{k}} \right\}_{\theta \in \mathbb{R}},$$

yielding that cone $(\mathcal{W}) \setminus \{0\} \supseteq \mathcal{W}_{\max} (\geq^*)$ and, in particular, cone $(\mathcal{W}) \setminus \{0\} = \mathcal{W}_{\max} (\geq^*)$. Since the functional $v \mapsto c(p, v)$ is quasiconcave over cone $(\mathcal{W}) \setminus \{0\}$ for all $p \in \Delta$, it is immediate to see that

$$V\left(p\right) = \min_{v \in \mathcal{W}} c\left(p, v\right) = \min_{v \in \operatorname{cone}(\mathcal{W}) \setminus \{0\}} c\left(p, v\right) \quad \forall p \in \Delta.$$

By Remark 4 and since $\mathcal{W} = \{v_i\}_{i=1}^n$ is a finite Cautious Utility representation, we have that \geq satisfies Axioms 1- 5. By Theorem 1 and its proof, $\mathcal{W}_{\max} (\geq')$ is a canonical Cautious

Utility representation for \geq . In particular, we have that

$$V(p) = \min_{v \in \mathcal{W}} c(p, v) = \min_{v \in \operatorname{cone}(\mathcal{W}) \setminus \{0\}} c(p, v) = \inf_{v \in \mathcal{W}_{\max}(\geq')} c(p, v) \quad \forall p \in \Delta$$

Since \geq' is the largest subrelation of \geq that satisfies the Independence axiom and $p \geq^* q$ implies $p \geq q$, we have that \geq^* is a subrelation of \geq' and $\mathcal{W}_{max}(\geq') \subseteq \mathcal{W}_{max}(\geq^*) =$ cone $(\mathcal{W}) \setminus \{0\}$. By contradiction, assume that $\mathcal{W}_{max}(\geq') \neq$ cone $(\mathcal{W}) \setminus \{0\}$. Since $\mathcal{W}_{max}(\geq')$ is a convex set closed with respect to strictly positive scalar multiplications, this implies that $\mathcal{W} \not\subseteq \mathcal{W}_{max}(\geq')$. If \mathcal{W} is a singleton, then \geq is Expected Utility and, in particular, \geq' is complete and coincides with \geq . This implies that $\mathcal{W} = \{v_1\}$ and $\mathcal{W}_{max}(\geq') =$ $\{\lambda v_1\}_{\lambda>0} = \text{cone}(\mathcal{W}) \setminus \{0\}$, a contradiction. Assume \mathcal{W} is not a singleton. Consider $\breve{v} \in \mathcal{W} \setminus \mathcal{W}_{max}(\geq')$. Since \mathcal{W} is essential, there exists $\bar{p} \in \Delta$ such that $\min_{v \in \mathcal{W}} c(\bar{p}, v) <$ $\min_{v \in \mathcal{W} \setminus \{\breve{o}\}} c(\bar{p}, v)$. Since $\mathcal{W} = \{v_i\}_{i=1}^n$ and $n \geq 2$, without loss of generality, we can set $\breve{v} = v_n \notin \mathcal{W}_{max}(\geq')$. In particular, we have that

$$\inf_{v \in \mathcal{W}_{\max}(\geq')} c(\bar{p}, v) = \min_{v \in \mathcal{W}} c(\bar{p}, v) = c(\bar{p}, v_n) < c(\bar{p}, v_i) \quad \forall i \in \{1, ..., n-1\}.$$
(20)

Consider a sequence $\{\hat{v}_m\}_{m\in\mathbb{N}} \subseteq \mathcal{W}_{\max}(\geq')$ such that $c(\bar{p}, \hat{v}_m) \downarrow \inf_{v\in\mathcal{W}_{\max}(\geq')} c(\bar{p}, v)$. By construction and since $\mathcal{W}_{\max}(\geq') \subseteq$ cone $(\mathcal{W}) \setminus \{0\}$, there exists a collection of scalars $\{\lambda_{m,i}\}_{m\in\mathbb{N},i\in\{1,\dots,n\}} \subseteq [0,\infty)$ such that $\hat{v}_m = \sum_{i=1}^n \lambda_{m,i}v_i$ for all $m \in \mathbb{N}$. Since \hat{v}_m is strictly increasing, we have that for each $m \in \mathbb{N}$ there exists $i \in \{1,\dots,n\}$ such that $\lambda_{m,i} > 0$. Define $\lambda_{m,\sigma} = \sum_{i=1}^n \lambda_{m,i} > 0$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ and for each $i \in \{1,\dots,n\}$ define also $\bar{\lambda}_{m,i} = \lambda_{m,i}/\lambda_{m,\sigma}$ as well as $\tilde{v}_m = \sum_{i=1}^n \bar{\lambda}_{m,i}v_i = \hat{v}_m/\lambda_{m,\sigma}$. Since $\lambda_{m,\sigma} > 0$ for all $m \in \mathbb{N}$, it is immediate to see that $c(\bar{p}, \tilde{v}_m) = c(\bar{p}, \hat{v}_m)$ for all $m \in \mathbb{N}$ and, in particular, $c(\bar{p}, \tilde{v}_m) \downarrow$ $\inf_{v\in\mathcal{W}_{\max}(\geq')} c(\bar{p}, v)$. For each $m \in \mathbb{N}$ denote by $\bar{\lambda}_m$ the \mathbb{R}^n vector whose *i*-th component is $\bar{\lambda}_{m,i}$. Since $\{\bar{\lambda}_m\}_{m\in\mathbb{R}}$ is a sequence in the \mathbb{R}^n simplex, there exists a subsequence $\{\bar{\lambda}_{m_i}\}_{i\in\mathbb{N}}$ such that $\bar{\lambda}_{m,i}.v_i \stackrel{\sigma(C(\mathbb{R}^k),\Delta)}{\longrightarrow} \sum_{i=1}^n \bar{\lambda}_i v_i = \tilde{v}$ where \tilde{v} is continuous, strictly increasing, and such that \tilde{v} (0) = 0. Moreover, for each $p, q \in \Delta$ we have that $p \geq' q$ implies $\mathbb{E}_p(\tilde{v}) \geq \mathbb{E}_q(\tilde{v})$, proving that $\tilde{v} \in \mathcal{W}_{\max}(\geq')$. Note that $\bar{\lambda}_n < 1$, otherwise, we would have that $v_n = \tilde{v} \in$ $\mathcal{W}_{\max}(\geq')$, a contradiction. By (20) and since $\bar{\lambda}_n < 1$ and the functional $v \mapsto c(p, v)$ is explicitly quasiconcave over co (W) for all $p \in \Delta$,²⁶ we have that

$$c\left(\bar{p}, v_{n}\right) < c\left(\bar{p}, \tilde{v}\right) = \lim_{l} c\left(\bar{p}, \tilde{v}_{m_{l}}\right) = \lim_{m} c\left(\bar{p}, \tilde{v}_{m}\right) = \inf_{v \in \mathcal{W}_{\max}(\geq')} c\left(\bar{p}, v\right) = c\left(\bar{p}, v_{n}\right),$$

a contradiction. It follows that $\mathcal{W}_{max}(\geq') = \operatorname{cone}(\mathcal{W}) \setminus \{0\}$ and, in particular, \mathcal{W} represents also \geq' . This implies that \mathcal{W} is canonical.

Proof of Proposition 4. We first prove the first part of the statement assuming \geq satisfies u-CPT, and then we will move to the additive case. Since u(0) = 0 and u is strictly increasing, it follows that there exists $\bar{t} > 0$ such that $[-\bar{t}, \bar{t}] \subseteq \text{Im } u$. Let $\Delta_0([0, \bar{t}])$ be the set of finitely supported probabilities over $[0, \bar{t}]$. Consider $\tilde{p} \in \Delta_0([0, \bar{t}])$. By definition, we have that there exist two unique collections $\{t_i\}_{i=1}^n \subseteq [0, \bar{t}]$ and $\{\lambda_i\}_{i=1}^n \subseteq [0, 1]$ such that supp p = $\{t_i\}_{i=1}^n, \sum_{i=1}^n \lambda_i = 1$, and $\tilde{p} = \sum_{i=1}^n \lambda_i \delta_{t_i}$. Without loss of generality, we can assume that $t_1 < ... < t_n$. We define $\tilde{V} : \Delta_0([0, \bar{t}]) \to \mathbb{R}$ by

$$\tilde{V}\left(\tilde{p}\right) = \sum_{j=1}^{n-1} \left(w^+ \left(\sum_{i=j}^n \lambda_i \right) - w^+ \left(\sum_{i=j+1}^n \lambda_i \right) \right) v\left(t_j\right) + w^+ \left(\lambda_n\right) v\left(t_n\right)$$

for all $\tilde{p} \in \Delta_0([0, \bar{t}])$. We next show that for each $\tilde{p} \in \Delta_0([0, \bar{t}])$ and for each $\tilde{t} \in [0, \bar{t}]$, if $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$, then $\tilde{V}(\lambda \tilde{p} + (1 - \lambda) \delta_{\tilde{t}}) = \tilde{V}(\delta_{\tilde{t}})$ for all $\lambda \in (0, 1)$. Consider $\tilde{p} \in \Delta_0([0, \bar{t}])$ and $\tilde{t} \in [0, \bar{t}]$ such that $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$. Given $\tilde{p} \in \Delta_0([0, \bar{t}])$, since $\{t_i\}_{i=1}^n \subseteq [0, \bar{t}] \subseteq \text{Im } u$, there exists $\{x_i\}_{i=1}^n \subseteq \mathbb{R}^k$ such that $u(x_i) = t_i$ for all $i \in \{1, ..., n\}$. Consider $p = \sum_{i=1}^n \lambda_i \delta_{x_i}$. It is immediate to see that $\tilde{V}(\tilde{p}) = V(p)$. Since \geq admits a Symmetric Cautious Utility representation, there exists $c \in \mathbb{R}$ such that $p \sim \delta_{ce_1}$. This implies that $V(p) = V(\delta_{ce_1})$ and, in particular, $u(ce_1) \in [0, \bar{t}]$. Moreover, since u and v are strictly increasing, we have that $u(ce_1) = \tilde{t} \in [0, \bar{t}]$ and $V(\delta_{ce_1}) = \tilde{V}(\delta_{\tilde{t}})$. By Remark 4 and since \geq admits a Symmetric Cautious Utility representation, we have that $\geq \text{satisfies M-NCI}$. This yields that $\lambda p + (1 - \lambda) \delta_{ce_1} \sim \delta_{ce_1}$ for all $\lambda \in (0, 1)$. This implies that

$$\tilde{V}\left(\lambda\tilde{p}+\left(1-\lambda\right)\delta_{\tilde{t}}\right)=V\left(\lambda p+\left(1-\lambda\right)\delta_{ce_{1}}\right)=V\left(\delta_{ce_{1}}\right)=\tilde{V}\left(\delta_{\tilde{t}}\right).$$

$$c(p,v_i) > c(p,v_h) \quad \forall i \in \{1,...,h-1\} \implies c\left(p,\sum_{i=1}^h \lambda_i v_i\right) > c(p,v_h).$$

²⁶Formally, see e.g. (Aliprantis and Border, 2006, p. 300), given $p \in \Delta$, for each $h \in \mathbb{N} \setminus \{1\}$, for each $\{v_l\}_{l=1}^h \subseteq \text{co}(\mathcal{W})$, and for each $\{\lambda_l\}_{l=1}^h \subseteq [0, 1]$ such that $\sum_{l=1}^h \lambda_l = 1$ and $\lambda_h < 1$

By Bell and Fishburn (2003, Theorem 1) applied to \tilde{V} , it follows that w^+ is the identity. The same proof, performed with $[-\bar{t}, 0]$ in place of $[0, \bar{t}]$ and w^+ replaced by w^- , yields that w^- is the identity. These two facts together allow us to conclude that $p \mapsto V(p) = \text{CPT}_{v,w^+,w^-}(p_u)$ is an Expected Utility functional with utility $v \circ u : \mathbb{R}^k \to \mathbb{R}$. We next assume that \geq admits an Additive CPT representation. As before consider $\bar{t} > 0$. Define $\Delta_0([0,\bar{t}])$ and \tilde{V} as before with v replaced by u_1 . For each $\tilde{p} \in \Delta_0([0,\bar{t}])$ define p in Δ to be the product measure $\tilde{p} \otimes \delta_{0} ... \otimes \delta_{0}$. It is immediate to see that $\tilde{V}(\tilde{p}) = V(p)$ for all $\tilde{p} \in \Delta_0([0,\bar{t}])$. As before, we can show that for each $\tilde{p} \in \Delta_0([0,\bar{t}])$ and for each $\tilde{t} \in [0,\bar{t}]$, if $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$, then $\tilde{V}(\lambda \tilde{p} + (1 - \lambda) \delta_{\tilde{t}}) = \tilde{V}(\delta_{\tilde{t}})$ for all $\lambda \in (0, 1)$. Consider $\tilde{p} \in \Delta_0([0,\bar{t}])$ and $\tilde{t} \in [0,\bar{t}]$ such that $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$. This implies that $V(p) = V(\delta_{\tilde{t}e_1})$, that is, $p \sim \delta_{\tilde{t}e_1}$. By Remark 4 and since \geq admits a Symmetric Cautious Utility representation, we have that \geq satisfies M-NCI. This yields that $\lambda p + (1 - \lambda) \delta_{\tilde{t}e_1} \sim \delta_{\tilde{t}e_1}$ for all $\lambda \in (0, 1)$. This implies that

$$\tilde{V}\left(\lambda\tilde{p}+\left(1-\lambda\right)\delta_{\tilde{t}}\right)=V\left(\lambda p+\left(1-\lambda\right)\delta_{\tilde{t}e_{1}}\right)=V\left(\delta_{\tilde{t}e_{1}}\right)=\tilde{V}\left(\delta_{\tilde{t}}\right).$$

By Bell and Fishburn (2003, Theorem 1) applied to \tilde{V} , it follows that w^+ is the identity. The same proof, performed with $[-\bar{t}, 0]$ in place of $[0, \bar{t}]$ and w^+ replaced by w^- , yields that w^- is the identity. This implies that \geq admits an Expected Utility representation with utility $u : \mathbb{R}^k \to \mathbb{R}$ defined by $u(x) = \sum_{i=1}^k u_i(x_i)$ for all $x \in \mathbb{R}^k$.

As for the second part of the statement, by Lemma 2 and since W is a finite essential Cautious Utility representation, we have that W is a canonical representation, that is, $W = \{v_i\}_{i=1}^n$ represents also \geq' . Since \geq is Expected Utility with utility $v \circ u$ (where in the linear case v is the identity and u is additively separable), we have that \geq' coincides with \geq , yielding that for each $i \in \{1, ..., n\}$ there exists $\lambda_i > 0$ such that $v_i = \lambda_i (v \circ u)$. This implies that $c (p, v_i) = c (p, v \circ u)$ for all $p \in \Delta$ and for all $i \in \{1, ..., n\}$. Since W is essential, this implies that W is a singleton. Since $W = \{v_1\}$ and W is odd, this implies that v_1 is odd and, in particular, \geq is loss neutral for risk and exhibits no endowment effect.

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