

STOCHASTIC IMPATIENCE AND THE SEPARATION OF TIME AND RISK PREFERENCES*

David Dillenberger[†] Daniel Gottlieb[‡] Pietro Ortoleva[§]

First version: October 2017

This version: April 6, 2020

Abstract

We study how the separation of time and risk preferences relates to a behavioral property that generalizes impatience to stochastic environments: Stochastic Impatience. We show that, within a broad class of models, Stochastic Impatience holds if and only if risk aversion is “not too high” relative to the inverse elasticity of intertemporal substitution. This result has implications for many known models. For example, for those of Epstein and Zin (1989) and Hansen and Sargent (1995), Stochastic Impatience is violated for all commonly used parameters.

Key words: Stochastic Impatience, Epstein-Zin preferences, Separation of Time and Risk preferences, Risk Sensitive preferences, Non-Expected Utility

JEL: D81, D90, G11, E7.

*We thank Markus Brunnermeier, Marco Di Maggio, Federico Echenique, Faruk Gul, Sergiu Hart, Jennifer La’O, Stephen Morris, Emi Nakamura, Wolfgang Pesendorfer, Jón Steinsson, Tomasz Strzalecki, Claudio Tebaldi, Gianluca Violante, Leeat Yariv, and Motohiro Yogo for their useful comments and suggestions. This work began while Dillenberger was visiting the Economics Department at Princeton University and Gottlieb was visiting the Economics Department at Harvard University. They are most grateful to these institutions for their hospitality.

[†]Department of Economics, University of Pennsylvania. Email: ddill@sas.upenn.edu.

[‡]London School of Economics. Email: d.gottlieb@lse.ac.uk.

[§]Department of Economics and Woodrow Wilson School, Princeton University. Email: pietro.ortoleva@princeton.edu.

1 Introduction

In the standard Expected Discounted Utility model, the inverse of the elasticity of intertemporal substitution (EIS) coincides with the coefficient of relative risk aversion. However, an enormous literature in macroeconomics, finance, and behavioral economics has pointed to the need to separate these two coefficients both on empirical and on conceptual grounds. Empirically, observations from lab experiments, longitudinal microdata, and the desire to fit macroeconomic and financial data require a higher coefficient of risk aversion than the inverse of EIS.¹ Conceptually, attitudes towards risk and towards intertemporal smoothing belong to different domains, and there is no compelling reason why they must be equal to each other. These observations led to models that separate risk attitudes from EIS: prominent examples include the CRRA-CES version of Epstein and Zin (1989) (henceforth EZ) and the Risk Sensitive preferences of Hansen and Sargent (1995) (henceforth HS). Given the fundamental role of risk aversion and EIS in economics, evaluating these models qualitatively and quantitatively is an issue of primary relevance.

In this paper, we show that a behavioral postulate that we call *Stochastic Impatience* imposes a bound on how high risk aversion can be relative to EIS. Consider the choice between the following two options:

- A. With equal probability, permanently increase consumption by either 20% starting today or by 10% starting next year;
- B. With equal probability, permanently increase consumption by either 10% starting today or by 20% starting next year.

Both options involve identical benefits, odds, and dates. However, option A pairs the highest increase (20%) with the earlier date, whereas B pairs it with the later date. What would, or should, an individual choose?

To the extent that an individual prefers higher payments sooner, she may prefer option A. One way to see this is by decomposing each alternative into two parts. Both A and B offer a basic lottery in which the individual receives an increase of 10% either today or next year, as well as a 50-50 chance of an additional increase of 10%. The only difference between them is when this additional payment is made: today in

¹For example, Barsky et al. (1997) study a cross section of American households and find that risk aversion and EIS are uncorrelated. See Bansal and Yaron (2004); Hansen et al. (2007); Barro (2009); Andreoni and Sprenger (2012); Nakamura et al. (2017) and references therein.

option A, next year in option B. An individual who prefers obtaining a payment sooner would prefer option A. Since this property is a version of impatience (preference for earlier payments) for risky environments, we call it Stochastic Impatience. Impatience and Stochastic Impatience are equivalent under Expected Discounted Utility, but not when time and risk preferences are separate. Our main result shows that under general conditions, Stochastic Impatience is violated whenever risk aversion is high enough for a fixed EIS.

We begin by considering the widely used CRRA-CES version of EZ. We show that Stochastic Impatience fails if the coefficient of risk aversion exceeds both the inverse of EIS and one. All applications of EZ that we are aware of use parameters in this range: assuming risk aversion above the inverse of EIS is the main reason to use EZ in the first place. For example, with the parameters used by Bansal and Yaron (2004), the individual would strictly prefer option B in the example above, violating Stochastic Impatience. For HS preferences, we show that Stochastic Impatience always fails if the range of utilities of consumption is large enough. For example, with the parameters of Tallarini Jr (2000), option B is again chosen.

We then move beyond EZ and HS, establishing a more general result. Consider any preference relation over lotteries over consumption streams.² Assume that (i) without risk, the preference relation coincides with discounted utility, so that it admits a representation $\sum D(t)u(x(t))$ for a decreasing D and an increasing u (note that little is assumed about D); and (ii) with risk, the preference relation satisfies the Expected Utility postulates. These assumptions hold for the vast majority of models, including EZ and HS.

We observe that, in the space we consider, any preference relation that satisfies these two assumptions can be represented by the expectation of $\phi(\sum D(t)u(x(t)))$, where ϕ is some increasing function over discounted utils. Known as the Kihlstrom-Mirman (KM) representation, this model is similar to Expected Discounted Utility except for the additional curvature ϕ . Since ϕ affects risk aversion but not EIS, it captures the separation between the two. Because any preference relation that satisfies our assumptions must admit such representation, the KM representation gives a convenient way to discuss the gap between time and risk attitudes.³

²Instead of considering dynamic preferences over temporal lotteries, as in EZ, it suffices to consider their static implications—how they evaluate lotteries over future consumption streams at a given point in time and for a given date of resolution of uncertainty. This contains all the information on the separation between time and risk preferences.

³The preferences implied in our setup by EZ and HS admit a KM representation with convenient

Our general result shows that Stochastic Impatience imposes a bound on how high risk aversion can be holding EIS fixed. In particular, we show that if ϕ is more concave than the log function, we can always construct a violation of Stochastic Impatience. Conversely, Stochastic Impatience holds if ϕ is less concave than log. This result unifies the findings described previously and shows that they hold in any model that satisfies our two basic assumptions.⁴ The results thus far were established in terms of properties of the functional forms. We also present a behavioral counterpart of our analysis using the notion of Residual Risk Aversion, which captures, directly in terms of preferences, the additional risk aversion relative to the level already implied by EIS.

There are two implications of our results on the incompatibility of Stochastic Impatience with the widespread assumption of risk aversion well above the inverse of EIS. First, if Stochastic Impatience is taken as an appealing property, our results highlight issues with current modeling, including all common parametrizations of leading approaches. In Section 5 we discuss possible ways to maintain Stochastic Impatience without sacrificing the fit of empirical data. Second, our results provide simple empirical tests. To verify that risk aversion exceeds the inverse of EIS as often assumed, it is sufficient to document a violation of Stochastic Impatience. This is a more direct and possibly simpler test than existing ones, which involve estimating the two parameters separately using multiple questions and assuming specific functional forms.

Our paper is not the first to point out implications of how the separation between time and risk is modeled. Epstein et al. (2014) argue that common parameterizations of EZ imply unrealistically strong preferences for early resolution of uncertainty. We show that these parameters also lead to a violation of Stochastic Impatience, a property distinct from preference over the timing of resolution of uncertainty. Bommier et al. (2017) show that many models that separate time and risk preferences, including common specifications of EZ, violate a Monotonicity property. The latter is unrelated to Stochastic Impatience: for example, EZ with both risk aversion and the inverse of EIS less than 1 satisfies Stochastic Impatience but not Monotonicity; conversely, HS always satisfies Monotonicity but violates Stochastic Impatience when the utility

functional forms (ϕ being CRRA or CARA, respectively). This of course does not mean that EZ and HS are special cases of KM—this is not the case in the full setup of temporal lotteries.

⁴In Appendix A we provide two extensions. First, to continuous time, showing that equivalent results hold. Second, to non-Expected Utility, where we show that exhibiting First-Order Risk Aversion implies violations of Stochastic Impatience.

range is large enough. Lastly, a companion paper, Dejarnette et al. (2020), studies theoretically and experimentally risk attitudes towards time lotteries, as well as their relationship with Stochastic Impatience.⁵

2 Framework

We study preferences on lotteries over consumption streams. Consider an interval of per-period consumption $C \subset \mathbb{R}_+$ and a set of dates $T = \{1, \dots, \bar{t}\}$, where \bar{t} is either finite or infinite.⁶ A consumption program $\mathbf{x} = (x(1), x(2), \dots, x(\bar{t}))$ yields consumption $x(t) \in C$ in period $t \in T$. Let $\mathcal{X} = C^T$ be the set of consumption programs and let Δ be the set of all probability measures over \mathcal{X} with finite support. Let \succsim be a complete and transitive preference relation over Δ .

With some abuse of notation, we refer to $\mathbf{x} \in \mathcal{X}$ both as the consumption program and as the lottery that gives it with certainty (i.e., the Dirac measure on \mathbf{x}) and refer to $x \in C$ as both the consumption x and the constant stream that gives it in every period. To further simplify notation, for any $t > 1$ we denote by $(c, t, x) \in C \times T \times C$ the stream that gives c in every period until $t - 1$ and x from t onwards, that is, $(c, c, \dots, \underbrace{c}_{t-1}, \underbrace{x}_t, x, \dots, \underbrace{x}_{\bar{t}})$.

We consider the static space of lotteries over streams, and not the more complex space of temporal lotteries used in Kreps and Porteus (1978) or EZ, because this subdomain is sufficient for our purposes and allows a simpler treatment that does not sacrifice generality. Any model over temporal lotteries induces preferences over Δ .⁷ Crucially, these static preferences contain all the information on the separation of time and risk preferences relevant for our analysis. In addition, restricting attention to this subdomain allows us to derive results for a richer class of models, independently of

⁵More specifically, in Dejarnette et al. (2020) we show in an experiment that the vast majority of individuals are not risk seeking over time lotteries, as implied by Expected Discounted Utility; but we also show that this is incompatible with Stochastic Impatience within a broad class of models and suggest generalizations to accommodate both.

⁶We focus on real-valued consumption and discrete time to simplify exposition and notation. We start from date one (and not zero) to allow the introduction of additional consumption at time zero that is not subject to uncertainty. Appendix A presents the extension to continuous time.

⁷Formally, lotteries over streams are embedded within temporal lotteries once we fix a start date and a time of resolution of uncertainty. For example, if we start from preferences over temporal lotteries, hold the time-zero consumption fixed and assume that uncertainty is resolved between periods zero and one, we obtain preferences over lotteries over streams (Δ).

how they are defined dynamically.

We now introduce the main property studied in this paper:

Definition 1 (Stochastic Impatience). The relation \succsim satisfies Stochastic Impatience if for any $t_1, t_2 \in T$ with $t_1 < t_2$, and any $c, x_1, x_2 \in C$ with $x_1 > x_2 > c$,

$$\frac{1}{2}(c, t_1, x_1) + \frac{1}{2}(c, t_2, x_2) \succsim \frac{1}{2}(c, t_2, x_1) + \frac{1}{2}(c, t_1, x_2). \quad (1)$$

Stochastic Impatience states that the individual prefers the lottery in which she either starts receiving higher payments earlier or lower payments later. It can be seen as a stochastic counterpart of the standard impatience property—the individual prefers higher payments sooner—to a stochastic environment. As mentioned in the introduction, an argument for its appeal can be made by decomposing each alternative into two parts. Note that both options offer the same basic lottery that pays x_2 starting at either t_1 or t_2 , along with an increment of $x_1 - x_2$. The difference between the two options is when the increment is paid: for the option on the left, it is paired with the earlier date t_1 ; while the option on the right pairs it with the later date t_2 . Insofar as the agent prefers to obtain it sooner, the option on the left may be preferred. We say that Stochastic Impatience fails if there exist $t_1 < t_2$ and $x_1 > x_2 > c$ such that (1) fails.

Stochastic Impatience is related to notions of multivariate risk aversion (e.g., Richard, 1975; see also Wakker et al., 2004) and, more generally, to supermodularity. What distinguishes it is the specification of one dimension as prize and the other as time, giving it a different appeal. To our knowledge, Stochastic Impatience is a new property—except that a version of it appears in a companion paper (Dejarnette et al., 2020).⁸ The most related, albeit different, condition appears in Bommier (2007), where multivariate risk aversion is applied to an intertemporal context with the goal of capturing correlation aversion.

In Expected Discounted Utility (henceforth EDU), impatience—a preference for earlier rewards—and Stochastic Impatience are equivalent:

Observation 1. Consider a preference relation \succsim that admits a representation $\mathbb{E}[\sum_T D(t)u(x(t))]$ for some strictly increasing u . Then, \succsim satisfies Stochastic Impatience if and only if D is weakly decreasing.

⁸Dejarnette et al. (2020) considers a setup of prize-date pairs instead of streams, and a version of Stochastic Impatience with payments in specific dates instead of changes to a stream.

While Definition 1 considers prizes that raise consumption permanently, different formulations are possible. For example, the increase in consumption could last a fixed number of periods, or even a single one. In Appendix A we show that, under general conditions, these formulations are equivalent to the one we consider.⁹

3 Stochastic Impatience in EZ and HS

3.1 Epstein-Zin preferences

We begin our formal analysis by considering the most widely used model that separates time and risk preferences: the CRRA-CES version of EZ. Let $C = \mathbb{R}_{++}$ and $\bar{t} = +\infty$. This model admits the following recursive representation:

$$V_t = \left\{ (1 - \beta) x(t)^{1 - \frac{1}{\psi}} + \beta [E_t (V_{t+1}^{1-\alpha})]^{\frac{1 - \frac{1}{\psi}}{1-\alpha}} \right\}^{\frac{1}{1 - \frac{1}{\psi}}} \quad (2)$$

where $\alpha \in \mathbb{R}_+ \setminus \{1\}$ is the coefficient of relative risk aversion and $\psi \in \mathbb{R}_+ \setminus \{1\}$ is the EIS. When $\alpha = \frac{1}{\psi}$, the model coincides with EDU.

The following result characterizes Stochastic Impatience in this model (all proofs appear in Appendix C):

Proposition 1. *Let \succsim be a preference relation that admits a representation as in (2). Then, it satisfies Stochastic Impatience if and only if either (i) $\alpha \leq \frac{1}{\psi}$, or (ii) $\alpha < 1$.*

In words, holding EIS ψ fixed, Stochastic Impatience imposes an upper bound on risk aversion α : below 1 or below $\frac{1}{\psi}$. To understand why this is the case, consider again the choice between lotteries A and B given in the introduction:

- A.** With equal probability, permanently increase consumption by either 20% starting today, or by 10% starting next year;
- B.** With equal probability, permanently increase consumption by either 10% starting today, or by 20% starting next year.

⁹Specifically, in Appendix A we show that under the conditions of Proposition 4 below, these properties are equivalent in continuous time and in discrete time as the time intervals become arbitrary small. In general, they are not equivalent in discrete time, although qualitative conclusions remain unchanged.

Stochastic Impatience prescribes that A should be preferred. Note also that of the four possible outcomes, the best is 20% starting today, the worst is 10% next year, while the other two are intermediate. Option A involves the best and the worst outcomes, while option B features the two ‘intermediate’ ones. Thus, option A has more spread in discounted utility but also has a higher expected discounted utility—since the higher discounting is applied to the smaller amount. When $\alpha = \frac{1}{\psi}$, i.e., with EDU, the agent cares only about the expected discounted utility, and thus strictly prefers option A. But when risk aversion is increased fixing EIS, the individual starts disliking the spread in discounted utilities, which favors option B. When risk aversion is high enough, this second effect prevails, leading the individual to prefer option B and violate Stochastic Impatience. Proposition 1 provides the exact condition for when this is the case: $\alpha > \max\{\frac{1}{\psi}, 1\}$.

The relevance of this result should be understood in light of the parameters used in the wide literature that adopts EZ. All applications that we are aware of assume $\alpha > \max\{\frac{1}{\psi}, 1\}$. Indeed, the possibility of incorporating a risk aversion greater than the inverse of EIS is a primary reason for adopting this model, and a relative risk aversion above one is also typically imposed to fit finance data. For example, Bansal et al. (2016) study annual data from 1930 to 2015 and estimate $\alpha = 9.67$ and $\psi = 2.18$ (see Example 1 below for other references). Proposition 1 shows that Stochastic Impatience fails in this range.

A different strand of the literature (that typically does not adopt EZ) has instead argued for EIS less than one.¹⁰ This implies $\frac{1}{\psi} > 1$. With this restriction, EZ necessarily violates Stochastic Impatience if $\alpha > \frac{1}{\psi}$.

Proposition 1 only shows that with given parameters there are instances in which Stochastic Impatience must be violated. We now provide an example of such violation.

Example 1. Consider again lotteries A and B described in the introduction and above. Stochastic Impatience implies that A is preferred to B. However, B is preferred adopting the EZ model with the parameters of many known papers: Bansal and Yaron (2004) ($\alpha = 10$, $\beta = 0.998$, $\psi = 1.5$), Bansal et al. (2016) ($\alpha = 9.67$, $\beta = .999$, $\psi = 2.18$), Nakamura et al. (2017) ($\alpha = 9$, $\beta = 0.99$, $\psi = 1.5$), and Colacito et al. (2018) ($\alpha = 10$, $\beta = 0.97$, $\psi = 1.1$).¹¹

¹⁰A discussion of this vast debate is outside our scope. See Campbell (1999), Attanasio and Weber (2010), Campbell (2003) and, for more recent contributions, Gruber (2013); Ortu et al. (2013); Crump et al. (2015); Best et al. (2017).

¹¹Even with EIS $\psi < 1$, option B is still preferred to A if risk aversion α is high enough. For

Finally, in EZ α and ψ also determine the preference over the timing of resolution of uncertainty: The agent prefers early (late) resolution whenever α is higher (smaller) than $\frac{1}{\psi}$ (Epstein and Zin, 1989). By restricting α and ψ , Stochastic Impatience thus indirectly restricts the attitude towards the timing of resolution of uncertainty, despite the fact that, in general, these two notions are conceptually independent. Imposing an upper bound on α limits the strength of the preference for early resolution of uncertainty. This links our results to those in Epstein et al. (2014), who argue that the parameters used in much of the literature imply levels of preference for early resolution of uncertainty that may be implausible. Here we show that these same parameters imply a violation of Stochastic Impatience.

To summarize, we have seen that with the parameters commonly used to fit macroeconomic and financial data, the CRRA-CES specification of EZ violates Stochastic Impatience.

3.2 Risk Sensitive preferences

We now consider the Risk Sensitive preferences of HS, which admit the recursive representation:

$$V_t = u(x(t)) - \beta \cdot \frac{1}{k} \cdot \ln(E[e^{-kV_{t+1}}]). \quad (3)$$

for $t = 1, 2, \dots$ (see also Bommier et al., 2017). The parameter k captures the degree of risk sensitivity; $k > 0$ increases aversion to risk compared to standard expected utility.

Proposition 2. *Suppose \succsim admits a representation as in (3). Stochastic impatience holds if and only if $\sup_{x \in C} \{u(x)\} - \inf_{x \in C} \{u(x)\} \leq -\frac{\ln(\beta)}{k\beta(1-\beta)}$.*

Under HS, Stochastic Impatience is violated if the utility range of prizes is large enough. This is necessarily the case if the utility function $u : C \rightarrow \mathbb{R}$ is unbounded above or below (such as with a CARA utility function and an unbounded consumption space). Otherwise, Stochastic Impatience requires the range of per-period utility ($\sup u(x) - \inf u(x)$) and the additional risk aversion k to be small enough.

example, with $\alpha = 10$ and $\beta = 0.998$, B is preferred for any $\psi > 0.2576$. With lower risk aversion, violations of Stochastic Impatience require higher prizes. For example, with the parameters of Nakamura et al. (2013) ($\alpha = 6.4$, $\beta = 0.967$, $\psi = 2$), a violation is observed with a low prize of 20% and a high prize of 30% of per-period consumption. With the parameters of Barro (2009) ($\alpha = 4$, $\beta = 0.948$, and $\psi = 2$), it is observed with a low prize of 35% and a high prize of 40%. With even less risk aversion, closer to one, violations of Stochastic Impatience require higher and higher prizes.

Like in the case of EZ, Proposition 2 only shows that violations exist, but does not specify which ones. The following is an example using an influential parameterization:

Example 2. Tallarini Jr (2000) considers HS preferences with $C = \mathbb{R}_{++}$, $u(x) = \ln(x)$, and $k = (1 - \beta)(\xi - 1)$, where ξ is the coefficient of relative risk aversion with respect to atemporal wealth gambles. First note that, since u is unbounded, Stochastic Impatience fails for any parameter values. Tallarini Jr (2000) shows that the model is able to match key moments in asset pricing for some $(\xi, \beta) \in [46, 180] \times [.991, .999]$. Consider again options A and B used in Example 1, where Stochastic Impatience posits that option A is preferred. Using HS with any of the parameters of Tallarini Jr (2000), option B is preferred instead.¹²

4 A general result

We have seen that Stochastic Impatience imposes a bound on the separation of time and risk preferences in two well-known models. We now show that this result holds more generally, proceeding as follows. First, we present a functional form that allows us to analyze in more generality the separation of time and risk preferences. Next, we introduce and discuss a behavioral counterpart of this separation, which we call Residual Risk Aversion. Finally, we present our results on the bounds imposed by Stochastic Impatience both in terms of the functional form representation and behaviorally.

4.1 The KM model

For simplicity, in the remainder we assume that the space of per-period consumption is a compact interval: $C = [\underline{x}, \bar{x}] \subset \mathbb{R}_+$.¹³ We focus on preferences that satisfy the following two assumptions.

Assumption 1 (Discounted Utility without risk). *There exist a strictly increasing and continuous function $u : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}_+$ and a strictly decreasing function $D : T \rightarrow$*

¹²Specifically, Stochastic Impatience fails if both prizes exceed a proportion $\beta^{\frac{1}{(1-\xi)(1-\beta)}} - 1$ of background consumption. With $\beta = .998$ and $\xi > 11.5$, we have $\beta^{\frac{1}{(1-\xi)(1-\beta)}} - 1 < 10\%$, so option B is preferred to option A.

¹³All results hold when C is unbounded above but the per-period utility over outcomes u is bounded (this guarantees that discounted sums are well-defined).

$[0, 1]$ such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$

$$\mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow \sum_{t \in T} D(t)u(x(t)) \geq \sum_{t \in T} D(t)u(y(t)).$$

Assumption 2 (Expected Utility). *The following hold:*

1. For all $p, q, r \in \Delta$ and $\lambda \in (0, 1)$, $p \succcurlyeq q \Leftrightarrow \lambda p + (1 - \lambda)r \succcurlyeq \lambda q + (1 - \lambda)r$;
2. For all $p, q, r \in \Delta$ with $p \succ q \succ r$, there exist $\lambda, \gamma \in (0, 1)$ such that $\lambda p + (1 - \lambda)r \succ q \succ \gamma p + (1 - \gamma)r$.

Assumption 1 posits that in the absence of risk, preferences can be modeled using Discounted Utility with generic discount function D . This is true for the vast majority of models and allows for many types of discounting (e.g., exponential, hyperbolic, quasi-hyperbolic). Assumption 2 posits the postulates of Expected Utility under risk, which are also satisfied by most models in the literature.

Assumptions 1 and 2 yield the following representation:

Observation 2. The relation \succcurlyeq satisfies Assumptions 1 and 2 if and only if there exist a strictly increasing and continuous $u : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}$, a strictly decreasing $D : T \rightarrow [0, 1]$, and a strictly increasing $\phi : u(C) \rightarrow \mathbb{R}$ such that \succcurlyeq is represented by

$$V(p) = \mathbb{E}_p \left[\phi \left(\frac{1}{\sum_T D(t)} \sum_T D(t)u(x(t)) \right) \right]. \quad (4)$$

Conditional on u and D , ϕ is unique up to a positive affine transformation.¹⁴

The representation in (4) is known as the Kihlstrom-Mirman (KM) representation, as it can be seen as an application of the multi-attribute function of Kihlstrom and Mirman (1974) to the context of time.¹⁵ Fixing D , the curvature of u captures EIS; risk aversion is captured by $\phi \circ u$. This means that ϕ is the additional curvature used only in the case of risk—the degree of separation between risk aversion and EIS.

¹⁴Assumption 1 guarantees a Discounted Utility representation without risk; Assumption 2 guarantees an Expected Utility representation with a given Bernoulli utility V . Since V and the Discounted Utility representation must be ordinally equivalent, there exists a strictly increasing function ϕ that makes them equal. For completeness, we present the formal proof in Appendix C.

¹⁵See, for example, Epstein and Zin (1989). This functional form is derived, in a different setup, in Dejarrette et al. (2020). A similar functional form is used in Edmans and Gabaix (2011); Garrett and Pavan (2011); Abdellaoui et al. (2017); Andersen et al. (2017); Apesteguia et al. (2019).

While the KM model is known, Observation 2 highlights that it is characterized by Assumptions 1 and 2, and thus provides a convenient functional form to study the static implications of most models used in the literature, including EZ and HS—as their static implications satisfy both our assumptions. Importantly, this representation contains all the information on the separation of time and risk preferences, in the function ϕ . The KM representation may take familiar shapes: below, we show that Expected Discounted Utility, EZ, and HS correspond to linear, CRRA, and CARA functions ϕ , respectively.¹⁶

Example 3 (Expected Discounted Utility). If ϕ is affine, \succsim can be represented by $\mathbb{E}[\sum_T D(t)u(x(t))]$, the standard case of EDU.

Example 4 (EZ with CRRA-CES). Fix a consumption for time zero and consider a preference \succsim over Δ generated by (2). Then, \succsim admits a KM representation, with $u(x) = \frac{x}{1-\frac{1}{\psi}}$, $D(t) = \beta^t$, and

$$\phi(z) = \begin{cases} z^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha < 1 < \psi \\ -(-z)^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha > 1 > \psi \\ -z^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } 1 < \alpha, 1 < \psi \\ (-z)^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha < 1, \psi < 1 \end{cases}. \quad (5)$$

Example 5 (HS). Fix a consumption for time zero and consider a preference relation \succsim over Δ generated by (3). Then, \succsim admits a KM representation with $D(t) = \beta^t$ and $\phi(x) = -\exp(-kx)$.

4.2 Residual Risk Aversion

We now introduce a behavioral notion which aims to capture the individual's additional risk aversion relative to the level already implied by EIS. We call this *Residual*

¹⁶Note that this does not imply that the KM model includes EZ and HS as special cases. Rather, it implies that the static implications of EZ and HS on Δ admit a KM representation. If applied dynamically on the larger space of temporal lotteries with different current-period consumption, the models are not nested (e.g., EZ and HS are dynamically consistent, while KM is not if applied dynamically without modification). But these differences are inconsequential in our setup. Still, this observation highlights that one can view EZ or HS as being composed of a *collection* of KM representations, where ϕ varies with the timing of resolution of uncertainty and with current consumption in order to allow for recursivity and dynamic consistency.

Risk Aversion. This property is an adaptation to our framework of Traeger (2014)'s notion of Intertemporal Risk Aversion.¹⁷ It is also naturally related to notions of multivariate risk aversion (Richard, 1975).

Definition 2. Let \succsim be a preference relation over Δ . We say that \succsim displays Residual Risk Aversion if for any $a, b, c, d, x \in [\underline{x}, \bar{x}]$ such that

$$(a, d, x, x, \dots) \sim (b, b, x, x, \dots) \quad \text{and} \quad (d, a, x, x, \dots) \sim (c, c, x, x, \dots)$$

we have

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succsim \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots).$$

We say that \succsim displays Residual Risk Seeking/Neutrality if the above is instead \preceq/\sim .

To understand this definition, suppose \succsim satisfies Assumption 1 with utility u and discount function D . If $(a, d, x, x, \dots) \sim (b, b, x, x, \dots)$ and $(d, a, x, x, \dots) \sim (c, c, x, x, \dots)$ it must be the case that

$$D(1)u(a)+D(2)u(d) = [D(1)+D(2)]u(b) \quad \text{and} \quad D(1)u(d)+D(2)u(a) = [D(1)+D(2)]u(c).$$

Thus, $u(a) + u(d) = u(b) + u(c)$. Notice also that we must have either $a > b > c > d$ or $a < b < c < d$, and that these depend only on how \succsim ranks streams without risk. Now suppose that we take a lottery that returns with equal chances the constant streams a or d ; and a lottery that returns with equal chances the constant streams b or c . If all risk aversion is included in the curvature of u , these two lotteries must be indifferent to one other, as $u(a) + u(d) = u(b) + u(c)$. But if the individual displays additional risk aversion not captured by the curvature of u , then the fact that we have $a > b > c > d$ or $a < b < c < d$ implies that she should prefer the lottery between the constant streams b and c , in which the utility spread is smaller.

We first link Residual Risk Aversion to the properties of the KM representation, showing that the curvature of ϕ is related to Residual Risk Aversion in a similar way to how the curvature of the Bernoulli utility function is related to risk aversion in standard Expected Utility theory:

Proposition 3. *Suppose \succsim admits a KM representation (ϕ, D, u) . Then, \succsim displays Residual Risk Aversion/Seeking/Neutrality if and only if ϕ is concave/convex/affine.*

¹⁷Similar to Proposition 3 and Observation 5 below, Traeger also gives a functional characterization of attitudes towards intertemporal risk aversion and a related comparative notion in his framework.

It follows from this result that Residual Risk Neutrality characterizes EDU given Assumptions 1 and 2, and that EZ allows one to introduce Residual Risk Aversion:

Observation 3 (EDU is characterized by Residual Risk Neutrality). Suppose \succsim satisfies Assumptions 1 and 2. Then, it admits an EDU representation if and only if it displays Residual Risk Neutrality.

Observation 4 (EZ preferences). Suppose \succsim admits a representation as in (2). Then \succsim displays Residual Risk Aversion/Neutrality/Seeking if and only if $\alpha \geq / = / \leq \frac{1}{\psi}$.

Recall that the CRRA-CES version of EZ displays a preference for early (a preference for late/neutrality towards) resolution of uncertainty if $\alpha > (< / =) \frac{1}{\psi}$. Therefore, in this model, \succsim displays Residual Risk Aversion (Seeking/Neutrality) if and only if, in the space of temporal lotteries, there is a preference for early (a preference for late/neutrality towards) resolution of uncertainty.

We can also introduce a comparative notion.

Definition 3. Consider two preference relations \succsim_1 and \succsim_2 over Δ . We say that \succsim_1 has more Residual Risk Aversion than \succsim_2 if they coincide on degenerate lotteries and if, for all $a > b > c > d$,

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succsim_2 \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots)$$

implies

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succsim_1 \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots).$$

The comparative notion above parallels standard ones for risk and ambiguity aversion (Epstein, 1999; Ghirardato and Marinacci, 2002): it only compares preferences that have the same ranking on risk-free streams; for any two such preferences, it defines one as more Residual Risk Averse than the other if it prefers the smallest “utility spread” at least as much as the other.

Since, as shown in Proposition 3, the curvature of ϕ determines Residual Risk Aversion in KM representations, we can obtain a comparative notion analogous to risk aversion in standard Expected Utility theory:

Observation 5. Let \succsim_1 and \succsim_2 be two preferences with KM representations (ϕ_1, u, D) and (ϕ_2, u, D) , respectively. Then, \succsim_1 has more Residual Risk Aversion than \succsim_2 if and only if there exist a strictly increasing and concave function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi_1 = f \circ \phi_2$.

4.3 General bounds by Stochastic Impatience

We now present our general result. As usual, we say that ϕ is more concave/convex than g if $\phi = f \circ g$ for some concave/convex f .

Proposition 4. *Let $C = [\underline{x}, \bar{x}]$, \succsim be a preference relation over Δ that satisfies Assumptions 1 and 2, and (ϕ, u, D) be a KM representation of \succsim .*

- (i) *If $\phi(v)$ is weakly less concave than $\ln(v - u(\underline{x}))$ for all $v \in u(C)$, then \succsim satisfies Stochastic Impatience;*
- (ii) *There exist $v_1, v_2 \in u(C)$ such that, if $\phi(v)$ is strictly more concave than $\ln(v - u(\underline{x}))$ for all $v \in [v_1, v_2]$, then \succsim violates Stochastic Impatience;*
- (iii) *There exists another preference relation \succsim' over Δ that satisfies Assumptions 1 and 2, has more Residual Risk Aversion than \succsim , and violates Stochastic Impatience;*
- (iv) *Let \succsim' be another preference relation over Δ that satisfies Assumptions 1 and 2 and has more Residual Risk Aversion than \succsim . Then:*
 - (a) *if \succsim violates Stochastic Impatience, so does \succsim' ;*
 - (b) *if \succsim' satisfies Stochastic Impatience, so does \succsim .*

Proposition 4 shows that Stochastic Impatience imposes an upper bound on how high risk aversion can be given EIS; it does so both in terms of the KM representation, in parts (i) and (ii), and in terms of our behavioral notion, in parts (iii) and (iv). It shows that, fixing EIS, increasing risk aversion will eventually lead to a violation of Stochastic Impatience. This result follows the same intuition of the one for EZ: Stochastic Impatience implies that the individual should prefer the lottery with a higher average but also higher spread in discounted utilities. Under EDU such spread does not matter. Under KM, however, concavity of ϕ induces the agent to be averse towards such spread. Proposition 4 shows that the ‘breaking-point’ where the aversion to spread overcomes the preference for higher average is when the curvature of ϕ is that of the log.

Note that parts (i) and (ii) together are not an if and only if statement: part (ii) requires ϕ to be more concave in an entire range $[v_1, v_2]$ —which depends on u and D —while the negation of part (i) requires it to hold at a (neighborhood of a) single point. This is due solely to the discreteness of time intervals: in Appendix A

we consider the model with continuous time and show that, in that case, Stochastic Impatience holds if and only if $\phi(v)$ is globally less concave than $\ln(v - u(\underline{x}))$.

5 Discussion

We have shown that within a broad class of models, Stochastic Impatience puts a bound on how high risk aversion can be relative to EIS, which rules out the parameters used in most applications of EZ and HS. This result has two implications.

First, it provides a simple way to test common parametrizations of existing models: they imply that Stochastic Impatience must be violated. Second, it implies that widespread approaches violate Stochastic Impatience. If we wish to preserve it, we need to either (i) keep as high risk aversion relative to EIS as typically assumed, but consider a class of models where our results do not hold; or, (ii) limit how high risk aversion can be and find different ways to match empirical patterns. We conclude the paper with a discussion of these two possibilities.

Beyond our assumptions. Our results were proved assuming Expected Utility under risk and Discounted Utility without Risk (with generic discounting). Do they hold more broadly?

There are two natural ways to extend beyond Expected Utility. First, one can adopt standard models of non-Expected Utility developed in the atemporal environment. In Appendix B we consider a very broad class that includes models with probability weighting and Disappointment Aversion.¹⁸ We show that (in a continuous time setting) these models always violate Stochastic Impatience whenever they exhibit First-Order Risk Aversion (Segal and Spivak, 1990)—as is the case in most specifications that use them.

Alternatively, one can consider models that maintain Expected Utility within each period, but violate it across periods. While this avenue has received little attention in the literature on non-Expected Utility, there are existing models that fit into this category. One example is the Dynamic Ordinal Certainty Equivalent model (Selden, 1978; Selden and Stux, 1978), where the individual first calculates the per-period certainty equivalents using one utility function, and then calculates their discounted

¹⁸Applications of these models have been suggested, starting in the original paper of Epstein and Zin (1989). See Backus et al. (2004) and references therein.

value using a different utility. This model satisfies Stochastic Impatience while allowing for a separation of time and risk preferences (Dejarnette et al., 2020; Selden and Wei, 2019). A discussion of the appeal of this specific functional form is beyond our scope, although some papers pointed to concerns with dynamic consistency and notions of monotonicity (Epstein and Zin, 1989; Chew and Epstein, 1990; Bommier et al., 2017).

Our other assumption is Discounted Utility without risk. Going beyond it requires dropping additive separability, as in models of habit formation or memorable consumption. Our results qualitatively extend, in the sense that for any such model Stochastic Impatience imposes a bound on how high risk aversion can be for a fixed EIS.¹⁹ However, this bound depends on the specifics of the non-separable preference considered.

Beyond high risk aversion. Models in finance and macroeconomics often require high risk aversion to capture the individuals’ unwillingness to take financial or similar risks. An alternative approach is to introduce other features into the model that have the same effect—e.g., ambiguity aversion/robustness concerns, incorrect beliefs about stock returns, or rational inattention. If these aspects are relevant but omitted from the model, risk aversion may be overestimated.

For example, if some of the equity premium is due to ambiguity aversion, incorporating it into the model may allow for much lower coefficients of risk aversion (Barillas et al., 2009). This would reduce the preference for early resolution of uncertainty to more realistic levels (Epstein et al., 2014) and allow for Stochastic Impatience: since the latter is based on objective lotteries, it is unaffected by ambiguity aversion. In general, any feature that reduces the individual’s willingness to undertake financial risk without modifying her attitude towards objective lotteries, as discussed in the surveys of Backus et al. (2004), Epstein and Schneider (2010), and Hansen and Sargent (2014), could provide a way to reconcile the empirical fit of the model with more moderate levels of risk aversion and—as we show in this paper—also with Stochastic Impatience.

¹⁹Even weakening Assumption 1 while maintaining Assumption 2, a concave enough ϕ makes the value of any lottery be arbitrarily close to the value of its worst outcome, thus generating a violation of Stochastic Impatience.

Appendix

A Continuous time and additional results

We now consider a continuous time formulation of the model from Section 4. Let $C = [\underline{x}, \bar{x}] \subset \mathbb{R}_+$ denote the space of per-period consumption. The set of dates is now $T = [0, \bar{t}]$, where $\bar{t} > 0$ may be $+\infty$. For each $\delta > 0$, (c, t, x, δ) denotes the consumption stream that returns $c \in C$ for $\tilde{t} \in T \setminus [t, t + \delta)$ and $x \in C$ for $\tilde{t} \in [t, t + \delta) \cap T$. Note that $(c, t, x, \bar{t} - t)$ is the stream that returns c until date t and x from t onward.

We consider preferences \succsim over lotteries over streams that can be represented by

$$V(p) = \mathbb{E}_p \left[\phi \left(\frac{\int_0^{\bar{t}} D(t) u(x(t)) dt}{\int_0^{\bar{t}} D(t) dt} \right) \right],$$

for lotteries p such that the integrals are well defined, where $u : C \rightarrow \mathbb{R}_+$ is continuous and strictly increasing, $D : T \rightarrow [0, 1]$ is continuous and strictly decreasing, and $\phi : [u(\underline{x}), u(\bar{x})] \rightarrow \mathbb{R}$ is strictly increasing.

Recall that Stochastic Impatience posits that individuals prefer to associate higher prizes to earlier dates. The analogous notion in continuous time is as follows:

Definition 4 (Stochastic Impatience'). The relation \succsim satisfies Stochastic Impatience' if for any $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2 < \bar{t}$ and any $c, x_1, x_2 \in C$ with $x_1 > x_2 > c$

$$\frac{1}{2}(c, t_1, x_1, \bar{t} - t_1) + \frac{1}{2}(c, t_2, x_2, \bar{t} - t_2) \succsim \frac{1}{2}(c, t_2, x_1, \bar{t} - t_2) + \frac{1}{2}(c, t_1, x_2, \bar{t} - t_1). \quad (6)$$

In the definition above, increases in consumption are permanent. We can also consider a stronger version that allows for prizes with a finite duration.

Definition 5 (Strong Stochastic Impatience). The relation \succsim satisfies Strong Stochastic Impatience if for any $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2 < \bar{t}$, any $\delta > 0$, and any $c, x_1, x_2 \in C$ with $x_1 > x_2 > c$

$$\frac{1}{2}(c, t_1, x_1, \delta) + \frac{1}{2}(c, t_2, x_2, \delta) \succsim \frac{1}{2}(c, t_2, x_1, \delta) + \frac{1}{2}(c, t_1, x_2, \delta). \quad (7)$$

We now show that, with continuous time, the two versions of Stochastic Impatience are equivalent and correspond to a bound to the concavity of ϕ .

Proposition 5. *Suppose time is continuous and let (ϕ, u, D) be a KM representation of \succsim . The following statements are equivalent:*

1. The relation \succsim satisfies Stochastic Impatience’;
2. The relation \succsim satisfies Strong Stochastic Impatience;
3. The function $\phi(v)$ is weakly less concave than $\ln(v - u(\underline{x}))$.

Note that this result is stronger than the one obtained in the first two parts of Proposition 4 in discrete time: here one concavity condition is both necessary and sufficient for Stochastic Impatience.²⁰

B Beyond Expected Utility

In this section we show that the tension between Stochastic Impatience and the separation of time and risk preferences does not rely on Expected Utility.

We extend beyond Expected Utility by assuming that preferences are at least *locally bilinear* at $\frac{1}{2}$. This generalization includes as special cases popular models such as those of probability weighting (Rank-Dependent Utility, Quiggin 1982, and Cumulative Prospect Theory, Tversky and Kahneman 1992) and Disappointment Aversion (Gul, 1991).²¹ In general, bilinearity holds if there is an increasing onto function $\pi : [0, 1] \rightarrow [0, 1]$, and a function f that evaluates (arbitrary) prizes, such that, for any x, y such that $f(x) > f(y)$, the prospect that yields x with probability λ and y otherwise is evaluated by $\pi(\lambda) f(x) + [1 - \pi(\lambda)] f(y)$. Since our goal is to be as general as possible, we only require preferences to be bilinear for equally likely binary lotteries ($\lambda = \frac{1}{2}$)—the *local bilinear* model (Dean and Ortoleva, 2017).²² Applying it to our setting, we obtain the following generalization of the KM model using the continuous time setup of Appendix A.

Definition 6. We say that \succsim admits a *local bilinear KM* representation if there exist strictly increasing and continuous $u : C \rightarrow \mathbb{R}_+$, a strictly decreasing $D : T \rightarrow [0, 1]$, a strictly increasing and differentiable $\phi : u(C) \rightarrow \mathbb{R}$, and $\pi(\frac{1}{2}) \in (0, 1)$, such that

²⁰In discrete time, Stochastic Impatience also does not imply Strong Stochastic Impatience.

²¹It also allows for generalizations of Rank-Dependent Expected Utility, e.g., the minimum from a set of probability distortions (Dean and Ortoleva, 2017). On the other hand, it does not encompass all models of risk preferences (e.g., it does not encompass Cautious Expected Utility, Cerreia-Vioglio et al. 2015).

²²This is a local specification of the bilinear (or biseparable) model of Ghirardato and Marinacci (2001) for objective risk. Here, preferences are not restricted to be bilinear in general, but only that there is some bilinear representation for 50/50 lotteries.

for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $p = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ with $\int_0^{\bar{t}} D(t)u(x(t))dt \geq \int_0^{\bar{t}} D(t)u(y(t))dt$ is evaluated according to:

$$V(p) = \pi\left(\frac{1}{2}\right)\phi\left(\frac{\int_0^{\bar{t}} D(t)u(x(t))dt}{\int_0^{\bar{t}} D(t)dt}\right) + \left[1 - \pi\left(\frac{1}{2}\right)\right]\phi\left(\frac{\int_0^{\bar{t}} D(t)u(y(t))dt}{\int_0^{\bar{t}} D(t)dt}\right).$$

It is easy to see that in a local bilinear KM representation Residual Risk Aversion can be achieved either by adding curvature to ϕ , as in the KM representation, or by adding non-Expected Utility and First-Order Risk Aversion (Segal and Spivak, 1990) by positing that $\pi(\frac{1}{2}) < \frac{1}{2}$, i.e., underweighting the best outcome.

Proposition 6. *Let \succsim be a preference relation over Δ that admits a local bilinear KM representation (u, D, ϕ, π) . If $\pi(\frac{1}{2}) < \frac{1}{2}$, then \succsim violates Stochastic Impatience'.*

The result above shows that, in continuous time, even if we go beyond Expected Utility by looking at the broad class of local bilinear models, displaying First-Order Risk Aversion *always* leads to violations of Stochastic Impatience, independently of the shape of ϕ . Intuitively, this derives from the fact that First-Order Risk Aversion implies extreme amounts of risk aversion in a neighborhood around certainty; and we have already seen how Stochastic Impatience is violated once risk aversion towards discounted utilities is high enough.

C Proofs

C.1 Proof of Observation 1

Using the representation, Stochastic Impatience holds if and only if

$$\frac{D(t_1)u(x_1) + D(t_2)u(x_2)}{2} + \sum_{t \notin \{t_1, t_2\}} D(t)u(c) \geq \frac{D(t_1)u(x_2) + D(t_2)u(x_1)}{2} + \sum_{t \notin \{t_1, t_2\}} D(t)u(c)$$

for all all $t_1 < t_2$, c , and $x_1 > x_2$. Rearrange this expression to obtain:

$$[D(t_1) - D(t_2)][u(x_1) - u(x_2)] \geq 0.$$

Since u is strictly increasing, this inequality holds if and only if $D(t_1) \geq D(t_2)$. ■

C.2 Proof of Proposition 1

Let $\rho \equiv \frac{1}{\psi}$ denote the inverse of EIS. For notational simplicity, we will work with ρ instead of ψ .

Since preferences are dynamically consistent, it suffices to look at lotteries in which the earliest payment is made in period 1. Consider a lottery that, with equal probability, either starts paying an increment of x in period 1 or starts paying an increment of y in period t : $\frac{1}{2}(c, 1, c + x) + \frac{1}{2}(c, t, c + y)$.

Note that \succsim satisfies Stochastic Impatience (SI) if:

$$\frac{1}{2}(c, 1, c + x) + \frac{1}{2}(c, t, c + y) \succsim \frac{1}{2}(c, 1, c + y) + \frac{1}{2}(c, t, c + x)$$

for any $c > 0$, any $x > y$, and any $t \in \{2, 3, \dots\}$. The proof will be given through a series of lemmas.

Lemma 1. *The value of lottery $\frac{1}{2}(c, 1, c + x) + \frac{1}{2}(c, t, c + y)$ is*

$$\left\{ (1 - \beta) c^{1-\rho} + \beta \left[\frac{(c + x)^{1-\alpha} + \{c^{1-\rho} + \beta^{t-1} [(c + y)^{1-\rho} - c^{1-\rho}]\}^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

Proof. Recall that in EZ, lotteries are evaluated using the recursion:

$$V_t = \left\{ (1 - \beta) x(t)^{1-\rho} + \beta [E_t (V_{t+1}^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}. \quad (8)$$

With 50% chance, the individual gets $c + x$ in all future periods, giving a continuation value of $V_1 = c + x$. With 50% chance, consumption equals c up to period $t - 1$, after which the individual gets $c + y$. Therefore, $V_t = c + y$. Proceeding backwards, we obtain:

$$V_{t-n} = \left\{ (1 - \beta) c^{1-\rho} (1 + \beta + \dots + \beta^{n-1}) + \beta^n (c + y)^{1-\rho} \right\}^{\frac{1}{1-\rho}},$$

for any $n = 1, \dots, t - 1$. In particular,

$$\begin{aligned} V_1 &= \left\{ (1 - \beta) c^{1-\rho} (1 + \beta + \dots + \beta^{t-2}) + \beta^{t-1} (c + y)^{1-\rho} \right\}^{\frac{1}{1-\rho}}, \\ &= \left\{ c^{1-\rho} + \beta^{t-1} [(c + y)^{1-\rho} - c^{1-\rho}] \right\}^{\frac{1}{1-\rho}}, \end{aligned}$$

where the second line uses the fact that $1 + \beta + \dots + \beta^{t-2} = \frac{1 - \beta^{t-1}}{1 - \beta}$.

Moving to period 0, we obtain:

$$V_0 = \left\{ (1 - \beta) c^{1-\rho} + \beta \left[\frac{(c+x)^{1-\alpha} + \{c^{1-\rho} + \beta^{t-1} [(c+y)^{1-\rho} - c^{1-\rho}]\}^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}} .$$

□

By the homotheticity of EZ preferences, we can, without loss of generality, take a background consumption of $c = 1$. Therefore, Stochastic Impatience holds if and only if, for all $z_H > z_L > 1$ and all $t \in \{2, 3, \dots\}$, we have

$$\begin{aligned} & \left\{ 1 - \beta + \beta \left[\frac{z_H^{1-\alpha} + [1 + \beta^{t-1} (z_L^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}} \\ & \geq \left\{ 1 - \beta + \beta \left[\frac{z_L^{1-\alpha} + [1 + \beta^{t-1} (z_H^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}} . \end{aligned} \quad (9)$$

Lemma 2. *Stochastic Impatience holds if and only if*

$$z^{\rho-\alpha} \geq [1 + \beta^{t-1} (z^{1-\rho} - 1)]^{\frac{\rho-\alpha}{1-\rho}} \beta^{t-1} \quad (10)$$

for all $z \geq 1$ and all $t = \{2, 3, \dots\}$.

Proof. Let $\Phi : [1, +\infty) \rightarrow \mathbb{R}$ be given by

$$\Phi(z) \equiv z^{1-\alpha} - [1 + \beta^{t-1} (z^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}} .$$

The proof has two parts. In the first part, we show that Stochastic Impatience holds if and only if:

- $\Phi'(z) \geq 0$ for all $z > 1$ if $\alpha < 1$.
- $\Phi'(z) \leq 0$ for all $z > 1$ if $\alpha > 1$.

To establish this result, we rearrange inequality (9) in each of 4 possible cases.

Case 1: $\alpha, \rho < 1$.

$$\begin{aligned} & z_H^{1-\alpha} + [1 + \beta^{t-1} (z_L^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}} \geq z_L^{1-\alpha} + [1 + \beta^{t-1} (z_H^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}} \\ \iff & z_H^{1-\alpha} - [1 + \beta^{t-1} (z_H^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}} \geq z_L^{1-\alpha} - [1 + \beta^{t-1} (z_L^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}} . \end{aligned}$$

Case 2: $\alpha, \rho > 1$.

$$\begin{aligned} z_H^{1-\alpha} + [1 + \beta^{t-1} (z_L^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}} &\leq z_L^{1-\alpha} + [1 + \beta^{t-1} (z_H^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}} \\ \iff z_H^{1-\alpha} - [1 + \beta^{t-1} (z_H^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}} &\leq z_L^{1-\alpha} - [1 + \beta^{t-1} (z_L^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}}. \end{aligned}$$

Case 3: $\alpha > 1 > \rho$.

$$z_H^{1-\alpha} - [1 + \beta^{t-1} (z_H^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}} \leq z_L^{1-\alpha} - [1 + \beta^{t-1} (z_L^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}}.$$

Case 4: $\rho > 1 > \alpha$.

$$z_H^{1-\alpha} - [1 + \beta^{t-1} (z_H^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}} \geq z_L^{1-\alpha} - [1 + \beta^{t-1} (z_L^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}}.$$

In the second part, we differentiate Φ to obtain:

$$\Phi'(z) = (1 - \alpha) \left\{ z^{-\alpha} - [1 + \beta^{t-1} (z^{1-\rho} - 1)]^{\frac{\rho-\alpha}{1-\rho}} \beta^{t-1} z^{-\rho} \right\}.$$

Since, by the previous result, Stochastic Impatience holds if $\Phi'(z) \geq 0$ when $\alpha < 1$ and $\Phi'(z) \leq 0$ if $\alpha > 1$, it follows that the term inside brackets must be weakly positive for all $z \geq 1$:

$$z^{-\alpha} \geq [1 + \beta^{t-1} (z^{1-\rho} - 1)]^{\frac{\rho-\alpha}{1-\rho}} \beta^{t-1} z^{-\rho}.$$

Multiplying both sides by $z^\rho > 0$, we obtain (10). □

We now use Lemma 2 to determine when Stochastic Impatience holds.

Lemma 3. *Stochastic Impatience if and only if either (i) $\alpha \leq \frac{1}{\psi}$, or (ii) $\alpha < 1$.*

Proof. The proof considers each of the six possible cases. We start with the two cases under which Stochastic Impatience fails:

Case 1. $\alpha > 1 > \rho$. To show that Stochastic Impatience fails in this case, note that after some algebraic manipulations, we can rewrite (10) as:

$$\frac{1 - \beta^{t-1}}{z^{1-\rho}} \geq \beta^{(t-1)\frac{1-\rho}{\alpha-\rho}} - \beta^{t-1}$$

for all $z \geq 1$. Note that the LHS converges to zero as $z \nearrow +\infty$. Moreover, the LHS is bounded away from zero since

$$\beta^{(t-1)\frac{1-\rho}{\alpha-\rho}} > \beta^{t-1} \iff (t-1)\frac{1-\rho}{\alpha-\rho} < t-1 \iff 1 < \alpha.$$

Therefore, there exists \bar{z} such that (10) fails for all $z > \bar{z}$, showing that Stochastic Impatience fails.

Case 2. $\alpha > \rho > 1$. To show that Stochastic Impatience fails, rearrange (10) as

$$\beta^{\frac{(t-1)(\rho-1)}{\alpha-\rho}} \geq (1 - \beta^{t-1}) z^{\rho-1} + \beta^{t-1}.$$

Note that, as $z \searrow 1$, the RHS converges to 1, whereas the LHS is always strictly less than 1 (since $\frac{(t-1)(\rho-1)}{\alpha-\rho} > 0$). Therefore, this inequality fails for z close to 1, showing that Stochastic Impatience fails.

We now turn to the cases where Stochastic Impatience holds:

Case 3. $1 > \alpha > \rho$. Rearranging (10), we find that Stochastic Impatience holds if and only if

$$[1 + \beta^{t-1} (z^{1-\rho} - 1)]^{\frac{\alpha-\rho}{1-\rho}} \geq z^{\alpha-\rho} \beta^{t-1}$$

for all $z > 1$. Rearrange this inequality as

$$\frac{1 - \beta^{t-1}}{z^{1-\rho}} + \beta^{t-1} \geq \beta^{\frac{(t-1)(1-\rho)}{\alpha-\rho}}.$$

Because LHS is decreasing in z and converges to β^{t-1} as $z \nearrow +\infty$, this condition holds for all $z > 1$ if and only if

$$\beta^{t-1} \geq \beta^{(t-1)\left(\frac{1-\rho}{\alpha-\rho}\right)},$$

which is true since $\rho < \alpha \leq 1$. Thus, Stochastic Impatience holds in this case.

Case 4. $1 < \alpha < \rho$. Rewrite (10) as

$$z^{\rho-1} \cdot (1 - \beta^{t-1}) + \beta^{t-1} \geq \beta^{\frac{(t-1)(\rho-1)}{\rho-\alpha}}.$$

Since the LHS is increasing in z , this inequality holds for all $z > 1$ if and only if it holds for $z = 1$:

$$1 \geq \beta^{\frac{(t-1)(\rho-1)}{\rho-\alpha}},$$

which is true because $\frac{(t-1)(\rho-1)}{\rho-\alpha} > 0$. Thus, Stochastic Impatience also holds in this case.

Case 5. $\alpha < \rho < 1$. Rewrite (10) as

$$\frac{1}{\beta^{\frac{(t-1)(1-\rho)}{\rho-\alpha}}} \geq \frac{1 - \beta^{t-1}}{z^{1-\rho}} + \beta^{t-1}.$$

Since the RHS is decreasing in z , this condition holds for all $z > 1$ if and only if it holds for $z = 1$:

$$\frac{1}{\beta^{\frac{(t-1)(1-\rho)}{\rho-\alpha}}} \geq 1 - \beta^{t-1} + \beta^{t-1} \iff 1 \geq \beta^{\frac{(t-1)(1-\rho)}{\rho-\alpha}},$$

which is true since $\frac{(t-1)(1-\rho)}{\rho-\alpha} \geq 0$ under the parameters above.

Case 6. $\alpha < 1 < \rho$. Notice that (10) can be simplified as

$$z^{\rho-1} (1 - \beta^{t-1}) + \beta^{t-1} \geq \beta^{(t-1)\left(\frac{\rho-1}{\rho-\alpha}\right)}.$$

Since the LHS is increasing in z , this condition holds for all $z > 1$ if and only if it holds for $z = 1$:

$$1 \geq \beta^{(t-1)\left(\frac{\rho-1}{\rho-\alpha}\right)},$$

which is true since $(t-1)\left(\frac{\rho-1}{\rho-\alpha}\right) > 0$. □

C.3 Proof of Proposition 2

The utility of the lottery $\frac{1}{2}(c, 1, c+x) + \frac{1}{2}(c, t, c+y)$ is

$$\frac{\exp\left(-k\frac{\beta}{1-\beta}u(c+x)\right) + \exp\left(-k \cdot \left(\frac{\beta-\beta^t}{1-\beta}u(c) + \frac{\beta^t}{1-\beta}u(c+y)\right)\right)}{2}.$$

Therefore, Stochastic Impatience fails if and only if there exist $x > y > c$ and $t \in \{2, 3, \dots\}$ such that

$$\begin{aligned} & \frac{\exp\left(-k\frac{\beta}{1-\beta}u(c+y)\right) + \exp\left(-k \cdot \left(u(c)\frac{\beta-\beta^t}{1-\beta} + u(c+x)\frac{\beta^t}{1-\beta}\right)\right)}{2} \\ & > \frac{\exp\left(-k\frac{\beta}{1-\beta}u(c+x)\right) + \exp\left(-k \cdot \left(u(c)\frac{\beta-\beta^t}{1-\beta} + u(c+y)\frac{\beta^t}{1-\beta}\right)\right)}{2} \end{aligned}$$

Rearrange this inequality as:

$$\begin{aligned} & \exp\left(-k\frac{\beta}{1-\beta}u(c+y)\right) - \exp\left(-k \cdot \left(u(c)\frac{\beta-\beta^t}{1-\beta} + u(c+y)\frac{\beta^t}{1-\beta}\right)\right) \\ & < \exp\left(-k\frac{\beta}{1-\beta}u(c+x)\right) - \exp\left(-k \cdot \left(u(c)\frac{\beta-\beta^t}{1-\beta} + u(c+x)\frac{\beta^t}{1-\beta}\right)\right). \end{aligned}$$

For simplicity, we will assume that u is a differentiable function, although it is immediate to generalize the argument for when it is not. Then, Stochastic Impatience fails if and only if there exist $x > c$ and $t \in \{2, 3, \dots\}$ such that

$$\frac{d}{dx} \left[\exp\left(-k\frac{\beta}{1-\beta}u(c+x)\right) - \exp\left(-k \cdot \left(u(c)\frac{\beta-\beta^t}{1-\beta} + u(c+x)\frac{\beta^t}{1-\beta}\right)\right) \right] > 0$$

Evaluating the derivative, the previous inequality becomes

$$\exp\left(-k\frac{\beta}{1-\beta}u(c+x)\right) < \beta^{t-1} \exp\left(-k\cdot\left(u(c)\frac{\beta-\beta^t}{1-\beta}+u(c+x)\frac{\beta^t}{1-\beta}\right)\right).$$

Since both sides are positive, we can take logs to obtain:

$$-k\frac{\beta}{1-\beta}u(c+x) < (t-1)\ln\beta - k\cdot\left(u(c)\frac{\beta-\beta^t}{1-\beta}+u(c+x)\frac{\beta^t}{1-\beta}\right),$$

which can be rearranged as:

$$u(c+x) - u(c) > \frac{(t-1)(1-\beta)}{\beta-\beta^t} \cdot \frac{-\ln\beta}{k}. \quad (11)$$

Therefore, Stochastic Impatience fails if and only if there exist $x > c$ and $t = \{2, 3, \dots\}$ such that (11) holds. For $t = 2$, condition (11) becomes

$$u(c+x) - u(c) > \frac{-\ln\beta}{\beta \cdot k}.$$

To complete the proof, we verify that this inequality holds for some t if and only if it holds for $t = 2$, that is:

$$\frac{-\ln\beta}{\beta \cdot k} \leq \frac{(t-1)(1-\beta)}{\beta-\beta^t} \cdot \frac{-\ln\beta}{k}$$

for all $t > 2$. To see this, rearrange the inequality above as

$$(t-1)(1-\beta) - 1 + \beta^{t-1} \geq 0. \quad (12)$$

At $\beta = 1$, both sides equal zero. The derivative of the expression on the RHS with respect to β is:

$$-(t-1) + (t-1)\beta^{t-2} = -(t-1)(1-\beta^{t-2}),$$

which is strictly negative for all $\beta \in [0, 1)$ and all $t > 2$. Thus, (12) holds for all $\beta \in [0, 1]$, concluding the proof. ■

C.4 Proof of Observation 2

For necessity, note that, when restricted to degenerate streams, the representation is a monotone transformation of $\sum_T D(t)u(x(t))$, so preferences must satisfy Assumption

1. Moreover, since risky lotteries are evaluated by taking expectations, preferences satisfy Assumption 2 as in Expected Utility Theory.

For sufficiency, by Assumption 1, there exist a strictly increasing and continuous $u : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}$ and a strictly decreasing $D : T \rightarrow [0, 1]$ such that \succsim restricted to \mathcal{X} is represented by

$$F^*(\mathbf{x}) := \sum_{t \in T} D(t)u(x(t)).$$

Applying a positive transformation, the same preference is also represented by

$$F(\mathbf{x}) := \frac{1}{\sum_T D(t)} \sum_T D(t)u(x(t)),$$

Note that $F(\mathcal{X}) = u(C)$. By Assumption 2, there exists $U : \mathcal{X} \rightarrow \mathbb{R}$ such that \succsim is represented by

$$V(p) := \mathbb{E}_p[U].$$

It follows that U and F represent the same preferences over \mathcal{X} , i.e., for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$,

$$U(\mathbf{x}) \geq U(\mathbf{y}) \Leftrightarrow \mathbf{x} \succsim \mathbf{y} \Leftrightarrow F(\mathbf{x}) \geq F(\mathbf{y}). \quad (13)$$

Therefore, there must exist an increasing $\phi : u(C) \rightarrow \mathbb{R}$ such that $U = \phi \circ F$.

We claim that ϕ must be strictly increasing. Suppose not. Then, there are $a, b \in u(C)$ with $a > b$ and $\phi(a) = \phi(b)$. Consider the streams \mathbf{x} and \mathbf{y} that return $u^{-1}(a)$ and $u^{-1}(b)$ each period, respectively. Since $a > b$ we must have $F(\mathbf{x}) = a > b = F(\mathbf{y})$. At the same time, since $\phi(a) = \phi(b)$, we have $U(\mathbf{x}) = \phi(F(\mathbf{x})) = \phi(a) = \phi(b) = \phi(F(\mathbf{y})) = U(\mathbf{y})$, violating (13).

The uniqueness claims follow from the same arguments as in the Expected Utility Theorem. \blacksquare

C.5 Proof of Proposition 4

Before presenting the proof, it is helpful to rewrite Stochastic Impatience in terms of the KM model. Stochastic Impatience holds if and only if, for all $x, y, c \in C$ with $x > y > c$ and all $t_1, t_2 \in T$ with $t_1 < t_2$,

$$\begin{aligned} & \phi \left(\frac{\sum_{\tilde{t} < t_1} D(\tilde{t})u(c) + \sum_{\tilde{t} \geq t_1} D(\tilde{t})u(x)}{\sum_{\tilde{t}=1}^{\tilde{t}} D(\tilde{t})} \right) + \phi \left(\frac{\sum_{\tilde{t} < t_2} D(\tilde{t})u(c) + \sum_{\tilde{t} \geq t_2} D(\tilde{t})u(y)}{\sum_{\tilde{t}=1}^{\tilde{t}} D(\tilde{t})} \right) \\ & \geq \phi \left(\frac{\sum_{\tilde{t} < t_1} D(\tilde{t})u(c) + \sum_{\tilde{t} \geq t_1} D(\tilde{t})u(y)}{\sum_{\tilde{t}=1}^{\tilde{t}} D(\tilde{t})} \right) + \phi \left(\frac{\sum_{\tilde{t} < t_2} D(\tilde{t})u(c) + \sum_{\tilde{t} \geq t_2} D(\tilde{t})u(x)}{\sum_{\tilde{t}=1}^{\tilde{t}} D(\tilde{t})} \right) \end{aligned}$$

Let $d(t) \equiv \frac{\sum_{\tilde{t} > t} D(\tilde{t})}{\sum_{\tilde{t}=1}^t D(\tilde{t})} \in (0, 1)$. Then, the previous condition becomes

$$\begin{aligned} & \phi(d(t_1)u(x) + (1 - d(t_1))u(c)) + \phi(d(t_2)u(y) + (1 - d(t_2))u(c)) \\ & \geq \phi(d(t_1)u(y) + (1 - d(t_1))u(c)) + \phi(d(t_2)u(x) + (1 - d(t_2))u(c)) \end{aligned}$$

for all $u(x), u(y), u(c) \in [u(\underline{x}), u(\bar{x})]$ with $u(x) > u(y) > u(c)$ and all $d(t_1), d(t_2) \in \left\{ \frac{\sum_{\tilde{t} > t} D(\tilde{t})}{\sum_{\tilde{t}=1}^t D(\tilde{t})} \right\}_{t \in T}$ with $d(t_1) > d(t_2)$.

Proof of Part (i). Let $\phi = g \circ \ln$ for some increasing and convex function g . It suffices to show that

$$\begin{aligned} & g(\ln(\lambda u(x) + (1 - \lambda)u(c))) + g(\ln(\gamma u(y) + (1 - \gamma)u(c))) \\ & \geq g(\ln(\lambda u(y) + (1 - \lambda)u(c))) + g(\ln(\gamma u(x) + (1 - \gamma)u(c))) \end{aligned} \quad (14)$$

for all $u(x) > u(y) > u(c) \geq u(\underline{x})$ and all $0 < \gamma < \lambda \leq 1$. Let

$$z \equiv \ln \left[\frac{(\lambda u(y) + (1 - \lambda)u(c))(\gamma u(x) + (1 - \gamma)u(c))}{\lambda u(x) + (1 - \lambda)u(c)} \right],$$

so that:

$$\begin{aligned} & \ln(\lambda u(x) + (1 - \lambda)u(c)) + z \\ & = \ln(\lambda u(y) + (1 - \lambda)u(c)) + \ln(\gamma u(x) + (1 - \gamma)u(c)). \end{aligned} \quad (15)$$

Because $\lambda u(x) + (1 - \lambda)u(c) > \max\{\lambda u(y) + (1 - \lambda)u(c), \gamma u(x) + (1 - \gamma)u(c)\}$, the equation above implies

$$z < \min\{\ln(\lambda u(y) + (1 - \lambda)u(c)), \ln(\gamma u(x) + (1 - \gamma)u(c))\}. \quad (16)$$

Combining (15) and (16) with the fact that g is convex, we have

$$\begin{aligned} & g(\ln(\lambda u(x) + (1 - \lambda)u(c))) + g(z) \\ & \geq g(\ln(\lambda u(y) + (1 - \lambda)u(c))) + g(\ln(\gamma u(x) + (1 - \gamma)u(c))). \end{aligned} \quad (17)$$

It can also be shown that

$$z < \ln(\gamma u(y) + (1 - \gamma)u(c)).^{23}$$

Then, since g is increasing, replacing $g(z)$ by $g(\ln(\gamma u(y) + (1 - \gamma)u(c)))$ in (17) implies (14).

²³To see this, observe that the ratio $\frac{\lambda u(x) + (1 - \lambda)u(c)}{\lambda u(y) + (1 - \lambda)u(c)}$ is strictly increasing in λ . Therefore, if we replace z by $\ln(\gamma u(y) + (1 - \gamma)u(c))$ in (15), we will have that the left hand side is strictly greater than the right hand side.

Proof of Part (ii). Pick any $y \in (\underline{x}, \bar{x})$, any $t_2 > 1$, and recall that $d(t) \equiv \frac{\sum_{\tau=t}^{\bar{t}} D(\tau)}{\sum_{\tau=1}^{\bar{t}} D(\tau)} \in (0, 1]$. Define $v_1 = u(\underline{x}) + d(t_2)(u(y) - u(\underline{x}))$ and $v_2 = u(\bar{x})$. Suppose $\phi(z)$ is more concave than $\ln(z - u(\underline{x}))$ in $[v_1, v_2]$. Then, there exists g strictly concave in this range such that $\phi(z) = g(\ln(z - u(\underline{x})))$. Note that we must have

$$\begin{aligned} \ln(d(1)[u(\bar{x}) - u(\underline{x})]) + \ln(d(t_2)[u(y) - u(\underline{x})]) \\ = \ln(d(1)[u(y) - u(\underline{x})]) + \ln(d(t_2)[u(\bar{x}) - u(\underline{x})]). \end{aligned}$$

Since $u(y) < u(\bar{x})$, $d(1) > d(t_2)$, and g is strictly concave, it follows that:

$$\begin{aligned} g(\ln(d(1)[u(\bar{x}) - u(\underline{x})])) + g(\ln(d(t_2)[u(y) - u(\underline{x})])) \\ < g(\ln(d(1)[u(y) - u(\underline{x})])) + g(\ln(d(t_2)[u(\bar{x}) - u(\underline{x})])), \end{aligned}$$

so that

$$\begin{aligned} \phi(d(1)[u(\bar{x}) - u(\underline{x})] + u(\underline{x})) + \phi(d(t_2)[u(y) - u(\underline{x})] + u(\underline{x})) \\ < \phi(d(1)[u(y) - u(\underline{x})] + u(\underline{x})) + \phi(d(t_2)[u(\bar{x}) - u(\underline{x})] + u(\underline{x})). \end{aligned}$$

Substituting back from the definition of $d(\cdot)$ and noting that $d(1) = 1$, we obtain:

$$\begin{aligned} \phi(u(\bar{x})) + \phi\left(\frac{\sum_{t=t_2}^{\bar{t}} D(t)u(y) + \sum_{t=1}^{t_2} D(t)u(\underline{x})}{\sum_{t=1}^{\bar{t}} D(t)}\right) \\ < \phi(u(y)) + \phi\left(\frac{\sum_{t=t_2}^{\bar{t}} D(t)u(\bar{x}) + \sum_{t=1}^{t_2} D(t)u(\underline{x})}{\sum_{t=1}^{\bar{t}} D(t)}\right). \end{aligned}$$

Hence $\frac{1}{2}\bar{x} + \frac{1}{2}(\underline{x}, t_2, y) \prec \frac{1}{2}y + \frac{1}{2}(\underline{x}, t_2, \bar{x})$, violating Stochastic Impatience.

Proof of part (iii). Consider a preference relation \succsim with KM representation (ϕ, u, D) . Define ϕ^c as the concave envelope of ϕ . By definition, ϕ^c is both concave and more concave than ϕ .

Now consider $\phi'(v) = \phi^c(g(\ln(v - u(\underline{x}))))$ for some concave $g : \mathbb{R} \rightarrow \mathbb{R}$. Consider the preference \succsim' induced by the KM representation (ϕ', u, D) . Because both ϕ^c and g are concave, $\phi'(v)$ is more concave than $\ln(v - u(\underline{x}))$: by Proposition 4, \succsim' violates Stochastic Impatience. Because $\ln(v - u(\underline{x}))$ and $g(v)$ are concave, ϕ' is more concave than ϕ^c , and thus of ϕ . By Observation 5, \succsim' has more Residual Risk Aversion than \succsim .

Proof of part (iv). The statement follows trivially from Proposition 4 and Observation 5. ■

C.6 Proof of Proposition 3

The proof will be presented in three lemmas.

Lemma 4. *Preferences are Residual Risk Averse (Seeking) if for all $v_1, v_2 \in [u(\underline{x}), u(\bar{x})]$,*

$$\phi(\gamma v_1 + (1 - \gamma) v_2) + \phi(\gamma v_2 + (1 - \gamma) v_1) \geq (\leq) \phi(v_1) + \phi(v_2) \quad (18)$$

where $\gamma \equiv \frac{D(1)}{D(1)+D(2)} \in (\frac{1}{2}, 1)$.

Proof. Note that by Definition 2 and the KM representation, preferences display Residual Risk Aversion whenever:

$$\begin{aligned} \phi\left(\frac{D(1)u(a) + D(2)u(d) + \sum_{t \notin \{1,2\}} D(t)u(x)}{\sum_T D(t)}\right) \\ = \phi\left(\frac{[D(1) + D(2)]u(b) + \sum_{t \notin \{1,2\}} D(t)u(x)}{\sum_T D(t)}\right) \end{aligned}$$

and

$$\begin{aligned} \phi\left(\frac{D(1)u(d) + D(2)u(a) + \sum_{t \notin \{1,2\}} D(t)u(x)}{\sum_T D(t)}\right) \\ = \phi\left(\frac{[D(1) + D(2)]u(c) + \sum_{t \notin \{1,2\}} D(t)u(x)}{\sum_T D(t)}\right) \end{aligned}$$

imply

$$\frac{\phi(u(b)) + \phi(u(c))}{2} \geq \frac{\phi(u(a)) + \phi(u(d))}{2}.$$

Since ϕ is strictly increasing, the first two equations can be simplified as:

$$u(b) = \frac{D(1)u(a) + D(2)u(d)}{D(1) + D(2)} \text{ and } u(c) = \frac{D(1)u(d) + D(2)u(a)}{D(1) + D(2)}.$$

Therefore, Residual Risk Aversion holds if and only if, for all a and all d,

$$\phi\left(\frac{D(1)u(a) + D(2)u(d)}{D(1) + D(2)}\right) + \phi\left(\frac{D(1)u(d) + D(2)u(a)}{D(1) + D(2)}\right) \geq \phi(u(a)) + \phi(u(d)). \quad (19)$$

Letting $\gamma \equiv \frac{D(1)}{D(1)+D(2)}$, $v_1 \equiv u(a)$, and $v_2 \equiv u(d)$ concludes the proof. □

Lemma 5. *Let (ϕ, u, D) be a KM representation of \succsim .*

- *If ϕ is discontinuous at any point $v \neq u(\underline{x})$, then \succsim is not Residual Risk Averse.*
- *If ϕ is discontinuous at any point $v \neq u(\bar{x})$, then \succsim is not Residual Risk Seeking.*

Proof. Suppose ϕ is discontinuous at $v > u(\underline{x})$. Let $\{h_n\} \searrow v$ be a non-increasing sequence that converges to v and $\{l_n\} \nearrow v$ be a non-decreasing sequence that converges to v . Let

$$\phi_+ := \lim_{n \rightarrow \infty} \phi(h_n) > \lim_{n \rightarrow \infty} \phi(l_n) = \phi_-.$$

For each n , let $u_{a_n} := h_n$ and $u_{d_n} := \frac{l_n - \gamma h_n}{1 - \gamma}$. Note that

$$u_{d_n} < \gamma u_{d_n} + (1 - \gamma) u_{a_n} < \gamma u_{a_n} + (1 - \gamma) u_{d_n} = l_n < v < h_n = u_{a_n}.$$

Since ϕ is bounded (by $\phi(u(\underline{x}))$ and $\phi(u(\bar{x}))$), we can assume that the sequences $\{\phi(\gamma u_{a_n} + (1 - \gamma) u_{d_n})\}$, $\{\phi(\gamma u_{d_n} + (1 - \gamma) u_{a_n})\}$, $\{\phi(u_{a_n})\}$, and $\{\phi(u_{d_n})\}$ are convergent (taking a subsequence if necessary). Therefore,

$$\lim_{n \rightarrow \infty} \phi(u_{d_n}) = \lim_{n \rightarrow \infty} \phi(\gamma u_{d_n} + (1 - \gamma) u_{a_n}) = \lim_{n \rightarrow \infty} \phi(\gamma u_{a_n} + (1 - \gamma) u_{d_n}) = \phi_-,$$

and

$$\lim_{n \rightarrow \infty} \phi(u_{a_n}) = \phi_+ > \phi_-.$$

Therefore, there exists \bar{n} such that for all $n > \bar{n}$,

$$\phi(\gamma u_{a_n} + (1 - \gamma) u_{d_n}) + \phi(\gamma u_{d_n} + (1 - \gamma) u_{a_n}) < \phi(u_{a_n}) + \phi(u_{d_n}),$$

which, by (18), shows that preferences are not Residual Risk Averse.

Next, suppose ϕ is discontinuous at $v < u(\bar{x})$. Let $\{h_m\} \searrow v$ be a non-increasing sequence that converges to v , let $\{l_m\} \nearrow v$ be an increasing sequence that converges to v . As before, let

$$\phi_+ := \lim_{m \rightarrow \infty} \phi(h_m) > \lim_{m \rightarrow \infty} \phi(l_m) = \phi_-,$$

where the limits exist by the Monotone Convergence Theorem.

For each m , take $u_{d_m} := l_m$ and take $u_{a_m} = \frac{h_m - \gamma l_m}{1 - \gamma}$. Note that

$$u_{a_m} > \gamma u_{a_m} + (1 - \gamma) u_{d_m} > \gamma u_{d_m} + (1 - \gamma) u_{a_m} = h_m > v > l_m = u_{d_m}.$$

As before (taking a subsequence if necessary), we have

$$\lim_{m \rightarrow \infty} \phi(u_{a_m}) = \lim_{m \rightarrow \infty} \phi(\gamma u_{a_m} + (1 - \gamma) u_{d_m}) = \lim_{m \rightarrow \infty} \phi(\gamma u_{d_m} + (1 - \gamma) u_{a_m}) = \phi_+,$$

and

$$\lim_{m \rightarrow \infty} \phi(u_{d_m}) = \phi_- < \phi_+.$$

Thus, there exists \bar{m} such that for all $m > \bar{m}$,

$$\phi(\gamma u_{a_m} + (1 - \gamma) u_{d_m}) + \phi(\gamma u_{d_m} + (1 - \gamma) u_{a_m}) > \phi(u_{a_m}) + \phi(u_{d_m}),$$

showing that preferences are not Residual Risk Seeking. \square

Lemma 6. *Let (ϕ, u, D) be a KM representation of \succsim .*

- \succsim is Residual Risk Averse if and only if ϕ is concave.
- \succsim is Residual Risk Seeking if and only if ϕ is convex.

Proof. To establish sufficiency, suppose, without loss of generality, that $v > w$, so that

$$v > \gamma v + (1 - \gamma)w > \gamma w + (1 - \gamma)v > w.$$

It follows from the definition of concavity (convexity) and inequality (18) that preferences are Residual Risk Averse (Seeking) if ϕ is concave (convex). We now establish necessity.

Suppose preferences are Residual Risk Averse. By the Lemma 5, ϕ must be continuous at any point $v > u(\underline{x})$. We need to show that ϕ is concave. Suppose not. Then, there exist $v, w \in [u(\underline{x}), u(\bar{x})]$ with $v > w$ and $\lambda \in (0, 1)$ such that

$$\lambda\phi(v) + (1 - \lambda)\phi(w) > \phi(\lambda v + (1 - \lambda)w). \quad (20)$$

Let $F : [0, 1] \rightarrow \mathbb{R}$ given by

$$F(\tilde{\lambda}) \equiv \phi\left(\tilde{\lambda}v + (1 - \tilde{\lambda})w\right) - \left[\tilde{\lambda}\phi(v) + (1 - \tilde{\lambda})\phi(w)\right],$$

and note that $F(\lambda) < 0$, while $F(1) = F(0) = 0$. Since ϕ can only be discontinuous at $u(\underline{x})$, $F(\tilde{\lambda})$ is continuous at all $\tilde{\lambda} > 0$. It is continuous at $\tilde{\lambda} = 0$ if either $w > u(\underline{x})$ or if ϕ is continuous at $u(\underline{x})$.

Let

$$L \equiv \left\{ \tilde{\lambda} \in [0, \lambda] : F(\tilde{\lambda}) \leq 0 \right\} \text{ and } H \equiv \left\{ \tilde{\lambda} \in [\lambda, 1] : F(\tilde{\lambda}) \geq 0 \right\}.$$

Let

$$l \equiv \sup L \text{ and } h \equiv \inf H.$$

Because $F(\tilde{\lambda})$ is continuous at all $\tilde{\lambda} > 0$ and $F(\lambda) < 0$, it follows that $l < \lambda < h$. Moreover, it follows from the definitions of the supremum and infimum that

$$F(\tilde{\lambda}) < 0 \quad \forall \tilde{\lambda} \in (l, h).$$

We claim that $F(l) = 0$. There are two cases to consider. If F is continuous at 0, then L is a compact and non-empty set ($0 \in L$), which implies that $F(l) = 0$. Suppose, instead, that F is discontinuous at 0, which can only happen if $w = u(\underline{x})$ and ϕ is discontinuous at $u(\underline{x})$. Because ϕ is increasing, the discontinuity must correspond to an upwards jump: $\phi(u(\underline{x})) < \lim_{z \searrow u(\underline{x})} \phi(z) =: \phi(u(\underline{x})_+)$. Since

$$\begin{aligned} \lim_{\tilde{\lambda} \searrow 0} F(\tilde{\lambda}) &= \lim_{\tilde{\lambda} \searrow 0} \left\{ \phi(\tilde{\lambda}v + (1 - \tilde{\lambda})w) - [\tilde{\lambda}\phi(v) + (1 - \tilde{\lambda})\phi(w)] \right\} \\ &= \phi(u(\underline{x})_+) - \phi(u(\underline{x})) > 0, \end{aligned}$$

and F is continuous at any $\tilde{\lambda} > 0$, there exists $\bar{\lambda} > 0$ such that $F(\tilde{\lambda}) > 0$ for all $\tilde{\lambda} \in (0, \bar{\lambda})$. Hence, again by continuity of F for $\tilde{\lambda} > 0$, $l \geq \bar{\lambda}$ by the definition of supremum. Therefore,

$$l \equiv \sup L = \sup \left\{ \tilde{\lambda} \in [\bar{\lambda}, \lambda] : F(\tilde{\lambda}) \leq 0 \right\}.$$

Because $\left\{ \tilde{\lambda} \in [\bar{\lambda}, \lambda] : F(\tilde{\lambda}) \leq 0 \right\}$ is compact ($F(\tilde{\lambda})$ is continuous for all $\tilde{\lambda} > 0$) and non-empty (λ belongs to it), it again follows that $F(l) = 0$.

Next, we show that $F(h) = 0$. Because $F(\lambda) < 0$ and $F(0) = 0$, (20) implies that $\lambda > 0$. Therefore, $F(\tilde{\lambda})$ is continuous at $[\alpha, 1]$, implying that H is a compact set. Because it is also non-empty ($1 \in H$), we must have $F(h) = 0$.

Substituting the definition of F , we have shown:

$$l\phi(v) + (1 - l)\phi(w) = \phi(lv + (1 - l)w), \quad (21)$$

$$h\phi(v) + (1 - h)\phi(w) = \phi(hv + (1 - h)w), \quad (22)$$

and

$$\tilde{\lambda}\phi(v) + (1 - \tilde{\lambda})\phi(w) > \phi(\tilde{\lambda}v + (1 - \tilde{\lambda})w) \quad (23)$$

for all $\tilde{\lambda} \in (l, h)$.

Let $w' \equiv lv + (1 - l)w$ and $v' \equiv hv + (1 - h)w$, so that $w < w' < v' < v$. Note that, for all $\lambda \in (0, 1)$, we have

$$\begin{aligned} \lambda w' + (1 - \lambda) v' &= \lambda [lv + (1 - l)w] + (1 - \lambda) [hv + (1 - h)w] \\ &= [\lambda l + (1 - \lambda) h] v + \{1 - [\lambda l + (1 - \lambda) h]\} w. \end{aligned} \quad (24)$$

Since $\lambda l + (1 - \lambda) h \in (l, h)$, we have

$$\begin{aligned} \phi(\lambda w' + (1 - \lambda) v') &= \phi([\lambda l + (1 - \lambda) h] v + \{1 - [\lambda l + (1 - \lambda) h]\} w) \\ &< [\lambda l + (1 - \lambda) h] \phi(v) + \{1 - [\lambda l + (1 - \lambda) h]\} \phi(w) \\ &= \lambda [l\phi(v) + (1 - l)\phi(w)] + (1 - \lambda) [h\phi(v) + (1 - h)\phi(w)] \\ &= \lambda \phi(lv + (1 - l)w) + (1 - \lambda) \phi(hv + (1 - h)w) \\ &= \lambda \phi(w') + (1 - \lambda) \phi(v') \end{aligned}$$

for all $\lambda \in (0, 1)$, where the first line uses (24), the second line uses equation (23), the third line follows from algebraic manipulations, the fourth line uses (21) and (22), and the last line substitutes the definitions of v' and w' . Since this inequality holds for all $\lambda \in (0, 1)$, in particular, it must hold for γ and $1 - \gamma$:

$$\phi(\gamma w' + (1 - \gamma) v') < \gamma \phi(w') + (1 - \gamma) \phi(v')$$

and

$$\phi(\gamma v' + (1 - \gamma) w') < \gamma \phi(v') + (1 - \gamma) \phi(w').$$

Combining these two inequalities, gives

$$\phi(\gamma w' + (1 - \gamma) v') + \phi(\gamma v' + (1 - \gamma) w') < \phi(w') + \phi(v'),$$

showing that Residual Risk Aversion fails. The proof for Residual Risk Seeking is analogous. \square

C.7 Proof of Observations 3, 4, and 5

Observation 3 is due to Proposition 3 and the fact that KM coincides with EDU if and only if ϕ is affine. Observation 4 follows from Proposition 3 and the KM representation of EZ given in Example 4. Observation 5 follows directly from Proposition 3. \blacksquare

C.8 Proof of Proposition 5

The proof of Proposition 5 will be presented through two lemmas. We first show that items 1 and 3 are equivalent to each other:

Lemma 7. *Stochastic Impatience' holds if and only if ϕ is less concave than $\ln(v - u(\underline{x}))$.*

Proof. The fact that Stochastic Impatience' holds if ϕ is less concave than $\ln(v - u(\underline{x}))$ follows by the same exact argument as in the discrete time case (see proof of part (i) of Proposition 4). We now show that Stochastic Impatience' fails if ϕ is not less concave than $\ln(v - u(\underline{x}))$.

For notational simplicity, let $h(x) \equiv u(x) - u(\underline{x})$. Let $\phi(v) = g \circ \ln$ for some increasing function $g : h(C) \rightarrow \mathbb{R}$. Suppose g is not convex. Then, there exist $H > L$ and $\epsilon > 0$ such that $L, H, (L + \epsilon), (H + \epsilon)$ are in the interior of $\ln(h(C))$ and

$$g(H + \epsilon) - g(H) < g(L + \epsilon) - g(L). \quad (25)$$

Let $y \equiv h^{-1}(\exp(H))$, $x \equiv h^{-1}(\exp(\epsilon + H))$, and $\lambda \equiv \exp(L - H) \in (0, 1)$, so that

$$H = \ln(h(y)), \quad H + \epsilon = \ln(h(x)), \quad \text{and} \quad L = \ln(\lambda h(y)).$$

Note that $x > y > \underline{x}$ (since H and L are in the interior of C). Moreover, a straightforward algebraic manipulation yields

$$L + \epsilon = \ln(\lambda h(y)) + \ln(h(x)) - \ln(h(y)) = \ln(\lambda h(x)).$$

Therefore, (25) becomes

$$g(\ln(h(x))) - g(\ln(h(y))) < g(\ln(\lambda h(x))) - g(\ln(\lambda h(y))),$$

which can be rearranged as:

$$g(\ln(u(x) - u(\underline{x}))) + g(\ln(\lambda(u(y) - u(\underline{x})))) < g(\ln(u(y) - u(\underline{x}))) + g(\ln(\lambda(u(x) - u(\underline{x})))),$$

violating Stochastic Impatience' with $c = \underline{x}$, $t_1 = 0$, and $t_2 \in (0, \bar{t})$ such that $\lambda = \frac{\int_{t_2}^{\bar{t}} D(t) dt}{\int_0^{\bar{t}} D(t) dt}$. \square

Lemma 7 established that $1 \iff 3$. Since Strong Stochastic Impatience becomes Stochastic Impatience' if we take $\delta = +\infty$, it follows that $2 \implies 1$. To conclude the proof, it suffices to show that $3 \implies 2$.

Lemma 8. *Suppose ϕ is less concave than $\ln(v - u(\underline{x}))$. Then, preferences satisfy Strong Stochastic Impatience.*

Proof. Use the representation to obtain the value of $\frac{1}{2}(c, t_1, x, \delta) + \frac{1}{2}(c, t_2, y, \delta)$:

$$\frac{\phi(\lambda_{t_1}^\delta u(c+x) + (1 - \lambda_{t_1}^\delta) u(c)) + \phi(\gamma_{t_2}^\delta u(c+y) + (1 - \gamma_{t_2}^\delta) u(c))}{2},$$

where $\lambda_{t_1}^\delta \equiv \frac{\int_{t \in [t_1, t_1 + \delta] \cap T} D(t) dt}{\int_0^t D(t) dt} > \gamma_{t_2}^\delta \equiv \frac{\int_{t \in [t_2, t_2 + \delta] \cap T} D(\tilde{t}) d\tilde{t}}{\int_0^t D(\tilde{t}) dt}$.

It suffices to show that if ϕ is less concave than $\ln(v - u(\underline{x}))$, then

$$\begin{aligned} & \phi(\lambda u(c+x) + (1 - \lambda) u(c)) + \phi(\gamma u(c+y) + (1 - \gamma) u(c)) \\ & \geq \phi(\lambda u(c+y) + (1 - \lambda) u(c)) + \phi(\gamma u(c+x) + (1 - \gamma) u(c)) \end{aligned} \quad (26)$$

for all $0 < \gamma < \lambda < 1$, $c < y < x$. Let $\phi(v) = g(\ln(v - u(\underline{x})))$ for some increasing and weakly convex g , and $u_x \equiv u(c+x) - u(\underline{x}) > u_y \equiv u(c+y) - u(\underline{x}) > u_c \equiv u(c) - u(\underline{x}) \geq 0$. Then equation (26) becomes:

$$\begin{aligned} & g(\ln(\lambda u_x + (1 - \lambda) u_c)) + g(\ln(\gamma u_y + (1 - \gamma) u_c)) \\ & \geq g(\ln(\lambda u_y + (1 - \lambda) u_c)) + g(\ln(\gamma u_x + (1 - \gamma) u_c)). \end{aligned} \quad (27)$$

We now show that the points on the LHS of (27) have both a higher mean and a higher spread than the points on the RHS. It will then follow from the fact that g is increasing and weakly convex that the inequality holds. To see that the points on the LHS have a higher mean, note that

$$\begin{aligned} \ln(\lambda u_x + (1 - \lambda) u_c) + \ln(\gamma u_y + (1 - \gamma) u_c) & \geq \ln(\lambda u_y + (1 - \lambda) u_c) + \ln(\gamma u_x + (1 - \gamma) u_c) \\ \iff [\lambda(1 - \gamma) - (1 - \lambda)\gamma] u_c u_x & \geq [(1 - \gamma)\lambda - (1 - \lambda)\gamma] u_c u_y \\ \iff (\lambda - \gamma) u_c (u_x - u_y) & \geq 0, \end{aligned}$$

which is true since $\lambda > \gamma$, $u_x > u_y$, and $u_c \geq 0$. To see that the points on the LHS have a higher spread, note that

$$\begin{aligned} \ln(\lambda u_x + (1 - \lambda) u_c) & > \max\{\ln(\lambda u_y + (1 - \lambda) u_c), \ln(\gamma u_x + (1 - \gamma) u_c)\} \\ & > \ln(\gamma u_y + (1 - \gamma) u_c). \end{aligned}$$

Thus, condition (27) holds. □

This concludes the proof of Proposition 5. ■

C.9 Proof of Proposition 6

Take $c = \underline{x}$ and, for simplicity, assume that $u(\underline{x}) = 0$.²⁴ Let $x_1 \in \text{int}(C)$ be such that $\phi'(u(x_1)) > 0$ (which exists because ϕ is differentiable and strictly increasing). Fix $t_1 = 0$ and $t_2 > 0$ and let x_2 be such that:

$$u(x_2) = u(x_1) \frac{\int_{t_2}^{\bar{t}} D(t) dt}{\int_0^{\bar{t}} D(t) dt}. \quad (28)$$

Note that $x_2 \in \text{int}(C)$ because $0 < u(x_2) < u(x_1)$. Thus, by construction, $(c, t_2, x_1) \sim (c, t_1, x_2)$.

We will show that, if t_2 is close enough to 0, then

$$\begin{aligned} & \pi \left(\frac{1}{2} \right) \phi(u(x_1)) + (1 - \pi \left(\frac{1}{2} \right)) \phi \left(u(x_2) \frac{\int_{t_2}^{\bar{t}} D(t) dt}{\int_0^{\bar{t}} D(t) dt} \right) \\ &= \pi \left(\frac{1}{2} \right) \phi(u(x_1)) + (1 - \pi \left(\frac{1}{2} \right)) \phi \left(u(x_1) \left(\frac{\int_{t_2}^{\bar{t}} D(t) dt}{\int_0^{\bar{t}} D(t) dt} \right)^2 \right) \\ & < \phi \left(u(x_1) \frac{\int_{t_2}^{\bar{t}} D(t) dt}{\int_0^{\bar{t}} D(t) dt} \right) \end{aligned}$$

where the equality above uses (28). First note that both sides equal $\phi(u(x_1))$ for $t_2 = 0$. We now show that the LHS falls faster than RHS when we increase t_2 slightly, generating a violation of Stochastic Impatience. By Leibniz's rule we have

$$\frac{\partial}{\partial t_2} \phi \left(u(x_1) \frac{\int_{t_2}^{\bar{t}} D(t) dt}{\int_0^{\bar{t}} D(t) dt} \right) \Big|_{t_2=0} = -\phi'(u(x_1)) \frac{D(0)}{\int_0^{\bar{t}} D(t) dt}$$

whereas

$$\begin{aligned} & \frac{\partial}{\partial t_2} \left[\pi \left(\frac{1}{2} \right) \phi(u(x_1)) + (1 - \pi \left(\frac{1}{2} \right)) \phi \left(u(x_1) \left(\frac{\int_{t_2}^{\bar{t}} D(t) dt}{\int_0^{\bar{t}} D(t) dt} \right)^2 \right) \right] \Big|_{t_2=0} \\ &= -(1 - \pi \left(\frac{1}{2} \right)) \phi'(u(x_1)) \frac{2D(0)}{\int_0^{\bar{t}} D(t) dt} \end{aligned}$$

²⁴Note that this is without loss as u can always be normalized as long as ϕ is appropriately adjusted to maintain the total utility value unchanged.

So we want to show that

$$\frac{-\phi'(u(x_1))D(0)}{\int_0^t D(t)dt} > - \left[1 - \pi \left(\frac{1}{2} \right) \right] \phi'(u(x_1)) \frac{2D(0)}{\int_0^t D(t)dt}$$

Since $\frac{\phi'(u(x_1))D(0)}{\int_0^t D(t)dt} > 0$, this is true if and only if $\pi(\frac{1}{2}) < \frac{1}{2}$. ■

References

- ABDELLAOUI, M., E. KEMEL, A. PANIN, AND F. VIEIDER (2017): “Take your Time or Take your Chance. On the Impact of Risk on Time Discounting,” Mimeo HEC.
- ANDERSEN, S., G. W. HARRISON, M. LAU, AND E. E. RUTSTROEM (2017): “Intertemporal Utility and Correlation Aversion,” *International Economic Review*, *forthcoming*.
- ANDREONI, J. AND C. SPRENGER (2012): “Risk Preferences Are Not Time Preferences,” *American Economic Review*, 102, 3357–76.
- APESTEGUIA, J., M. Á. BALLESTER, A. GUTIERREZ, ET AL. (2019): “Random models for the joint treatment of risk and time preferences,” *Barcelona GSE Working Paper: 1117*.
- ATTANASIO, O. P. AND G. WEBER (2010): “Consumption and saving: models of intertemporal allocation and their implications for public policy,” *Journal of Economic literature*, 48, 693–751.
- BACKUS, D. K., B. R. ROUTLEDGE, AND S. E. ZIN (2004): “Exotic preferences for macroeconomists,” *NBER Macroeconomics Annual*, 19, 319–390.
- BANSAL, R., D. KIKU, AND A. YARON (2016): “Risks for the long run: Estimation with time aggregation,” *Journal of Monetary Economics*, 82, 52–69.
- BANSAL, R. AND A. YARON (2004): “Risks For the Long Run: A Potential Resolution of Asset Pricing Puzzles,” *Journal of Finance*, 59, 1481 – 1509.
- BARILLAS, F., L. P. HANSEN, AND T. J. SARGENT (2009): “Doubts or variability?” *journal of economic theory*, 144, 2388–2418.
- BARRO, R. J. (2009): “Rare disasters, asset prices, and welfare costs,” *American Economic Review*, 99, 243–64.
- BARSKY, R. B., F. T. JUSTER, M. S. KIMBALL, AND M. D. SHAPIRO (1997): “Preference parameters and behavioral heterogeneity: An experimental approach in the health and retirement study,” *The Quarterly Journal of Economics*, 112, 537–579.
- BEST, M. C., J. CLOYNE, E. ILZETZKI, AND H. KLEVEN (2017): “Estimating the Elasticity of Intertemporal Substitution Using Mortgage Notches,” Tech. rep., mimeo LSE.

- BOMMIER, A. (2007): “Risk Aversion, Intertemporal Elasticity of Substitution and Correlation Aversion,” *Economics Bulletin*, 4, 1–8.
- BOMMIER, A., A. KOCHOV, AND F. LE GRAND (2017): “On Monotone Recursive Preferences,” *Econometrica*, 85, 1433–1466.
- CAMPBELL, J. Y. (1999): “Asset prices, consumption, and the business cycle,” in *Handbook of macroeconomics*, ed. by J. B. Taylor and H. Uhlig, Elsevier, vol. 1, 1231–1303.
- (2003): “Consumption-based asset pricing,” in *Handbook of the Economics of Finance*, ed. by G. Constantinides, M. Harris, and R. M. Stulz, Elsevier, vol. 1, 803–887.
- CERREIA-VIOGLIO, S., D. DILLENBERGER, AND P. ORTOLEVA (2015): “Cautious Expected Utility and the Certainty Effect,” *Econometrica*, 83, 693–728.
- CHEW, S. H. AND L. G. EPSTEIN (1990): “Unexpected utility preferences in a temporal framework with an application to consumption-savings behaviour,” *Journal of Economic Theory*, 50, 54–81.
- COLACITO, R., M. M. CROCE, S. HO, AND P. HOWARD (2018): “BKK the EZ Way: International Long-Run Growth News and Capital Flows,” *American Economic Review* (forthcoming).
- CRUMP, R. K., S. EUSEPI, A. TAMBALOTTI, AND G. TOPA (2015): “Subjective intertemporal substitution,” Federal Reserve Bank of New York Staff Report No. 734.
- DEAN, M. AND P. ORTOLEVA (2017): “Allais, Ellsberg, and Preferences for Hedging,” *Theoretical Economics*, 12, 377–424.
- DEJARNETTE, P., D. DILLENBERGER, D. GOTTLIEB, AND P. ORTOLEVA (2020): “Time Lotteries and Stochastic Impatience,” *Econometrica*, 88, 619–656.
- EDMANS, A. AND X. GABAIX (2011): “Tractability in incentive contracting,” *Review of Financial Studies*, 24, 2865–2894.
- EPSTEIN, L. AND S. ZIN (1989): “Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework,” *Econometrica*, 57, 937–969.
- EPSTEIN, L. G. (1999): “A definition of uncertainty aversion,” *Review of Economic Studies*, 66, 579–608.
- EPSTEIN, L. G., E. FARHI, AND T. STRZALECKI (2014): “How Much Would You Pay to Resolve Long-Run Risk?” *American Economic Review*, 104, 2680–97.
- EPSTEIN, L. G. AND M. SCHNEIDER (2010): “Ambiguity and asset markets,” *Annual Review of Financial Economics*, 2, 315–346.
- GARRETT, D. AND A. PAVAN (2011): “Dynamic Managerial Compensation: On the Optimality of Seniority-based Schemes,” Mimeo Northwestern University.

- GHIRARDATO, P. AND M. MARINACCI (2001): “Risk, ambiguity, and the separation of utility and beliefs,” *Mathematics of Operations Research*, 864–890.
- (2002): “Ambiguity Made Precise: A Comparative Foundation,” *Journal of Economic Theory*, 102, 251–289.
- GRUBER, J. (2013): “A Tax-Based Estimate of the Elasticity of Intertemporal Substitution,” *Quarterly Journal of Finance*, 3.
- GUL, F. (1991): “A theory of disappointment aversion,” *Econometrica*, 59, 667–686.
- HANSEN, L. P., J. HEATON, J. LEE, AND N. ROUSSANOV (2007): “Intertemporal substitution and risk aversion,” *Handbook of econometrics*, 6, 3967–4056.
- HANSEN, L. P. AND T. J. SARGENT (1995): “Discounted linear exponential quadratic gaussian control,” *IEEE Transactions on Automatic control*, 40, 968–971.
- (2014): *Uncertainty within economic models*, vol. 6, World Scientific.
- KIHLSTROM, R. E. AND L. J. MIRMAN (1974): “Risk aversion with many commodities,” *Journal of Economic Theory*, 8, 361–388.
- KREPS, D. AND E. PORTEUS (1978): “Temporal resolution of uncertainty and dynamic choice theory,” *Econometrica*, 46, 185–200.
- NAKAMURA, E., D. SERGEYEV, AND J. STEINSSON (2017): “Growth-rate and uncertainty shocks in consumption: Cross-country evidence,” *American Economic Journal: Macroeconomics*, 9, 1–39.
- NAKAMURA, E., J. STEINSSON, R. BARRO, AND J. URSÚA (2013): “Crises and recoveries in an empirical model of consumption disasters,” *American Economic Journal: Macroeconomics*, 5, 35–74.
- ORTU, F., A. TAMONI, AND C. TEBALDI (2013): “Long-run risk and the persistence of consumption shocks,” *The Review of Financial Studies*, 26, 2876–2915.
- QUIGGIN, J. (1982): “A theory of anticipated utility,” *Journal of Economic Behavior & Organization*, 3, 323–343.
- RICHARD, S. F. (1975): “Multivariate risk aversion, utility independence and separable utility functions,” *Management Science*, 22, 12–21.
- SEGAL, U. AND A. SPIVAK (1990): “First Order versus Second Order Risk Aversion,” *Journal of Economic Theory*, 51, 111–125.
- SELDEN, L. (1978): “A New Representation of Preferences over “certain x uncertain” Consumption Pairs: The “Ordinal Certainty Equivalent” Hypothesis,” *Econometrica*, 1045–1060.

- SELDEN, L. AND I. E. STUX (1978): “Consumption Trees, OCE Utility and the Consumption/Savings Decision,” Mimeo Columbia University.
- SELDEN, L. AND X. WEI (2019): “Aversion to Intertemporal Substitution and Risk in Time Lotteries and Private Equity Investments,” Mimeo Columbia University.
- TALLARINI JR, T. D. (2000): “Risk-sensitive real business cycles,” *Journal of monetary Economics*, 45, 507–532.
- TRAEGER, C. (2014): “Capturing Intrinsic Risk Attitude,” Mimeo, University of California Berkeley.
- TVERSKY, A. AND D. KAHNEMAN (1992): “Advances in prospect theory: cumulative representation of uncertainty,” *Journal of Risk and Uncertainty*, 5, 297–323.
- WAKKER, P. P., S. J. JANSEN, AND A. M. STIGGELBOUT (2004): “Anchor levels as a new tool for the theory and measurement of multiattribute utility,” *Decision Analysis*, 1, 217–234.