Online Appendix

This appendix includes all the missing proofs and the ancillary facts used in the main body of the paper. We begin with a section on facts instrumental for Theorem 1 and Proposition 5.

Foundation

Recall the definition of $\succcurlyeq'$ in Section 5, that is,

$$p \succcurlyeq' q \overset{\text{def}}{\iff} \lambda p + (1-\lambda) r \succcurlyeq \lambda q + (1-\lambda) r \qquad \forall \lambda \in (0,1], \forall r \in \Delta.$$

The goal of this section is to provide a Multi-Expected Utility representation for $\succcurlyeq'$.

Lemma 1. *Let $\succcurlyeq$ be a binary relation on Δ that satisfies Weak Order. The following statements are true:*

1. *The relation $\succcurlyeq$ satisfies M-NCI if and only if for each $p \in \Delta$ and for each $m \in \mathbb{R}$*

$$p \succcurlyeq \delta_{me_1} \implies p \succcurlyeq' \delta_{me_1}. \qquad \text{(Equivalently } p \not\succcurlyeq' \delta_{me_1} \implies \delta_{me_1} \succ p.\text{)}$$

2. *If $\succcurlyeq$ satisfies Monotonicity, then for each $x, y \in \mathbb{R}^k$*

$$x > y \implies \delta_x \succ' \delta_y. \tag{12}$$

3. *If $\succcurlyeq$ satisfies Monetary equivalent, then for each $x, y \in \mathbb{R}^k$ there exists $m \in \mathbb{R}_+$ such that*

$$\delta_{y+me_1} \succcurlyeq' \delta_x \succcurlyeq' \delta_{y-me_1}. \tag{13}$$

Proof. All three points follow from the definition of $\succcurlyeq'$ and M-NCI, Monotonicity, and Monetary equivalent, respectively. ■

Aumann Utilities and Multi-Expected Utility Representations

In this section, in our formal results, we consider a binary relation $\succcurlyeq^*$ over Δ such that

$$p \succcurlyeq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W} \tag{14}$$

where $\mathcal{W} \subseteq C\left(\mathbb{R}^k\right)$. Recall that a function $v \in C\left(\mathbb{R}^k\right)$ is an Aumann utility if and only if

$$p \succ^* q \implies \mathbb{E}_p(v) > \mathbb{E}_q(v) \text{ and } p \sim^* q \implies \mathbb{E}_p(v) = \mathbb{E}_q(v).$$

We denote by e the vector whose components are all 1s. We endow $C\left(\mathbb{R}^k\right)$ with the distance $d : C\left(\mathbb{R}^k\right) \times C\left(\mathbb{R}^k\right) \to [0, \infty)$ defined by

$$d(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \min\left\{\max_{x \in [-ne, ne]} |f(x) - g(x)|, 1\right\} \quad \forall f, g \in C\left(\mathbb{R}^k\right).$$

It is routine to show that $\left(C\left(\mathbb{R}^k\right), d\right)$ is separable.[25] Moreover, if $\{f_m\}_{m\in\mathbb{N}} \subseteq C\left(\mathbb{R}^k\right)$ is such that $f_m \xrightarrow{d} f$, then $\{f_m\}_{m\in\mathbb{N}}$ converges uniformly to f on each compact subset of $\mathbb{R}^k$.

Proposition 7. *If $\succcurlyeq^*$ is as in (14) and such that*

$$x > y \implies \delta_x \succ^* \delta_y, \tag{15}$$

then $\succcurlyeq^*$ admits a strictly increasing Aumann utility.

Proof. By (14), observe that $x > y$ implies $v(x) \geq v(y)$ for all $v \in \mathcal{W}$. This implies that each $v \in \mathcal{W}$ is increasing. By Aliprantis and Border (2006, Corollary 3.5), there exists a countable d-dense subset D of $\mathcal{W}$. Clearly, we have that

$$p \succcurlyeq^* q \implies \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in D. \tag{16}$$

Vice-versa, consider $p, q \in \Delta$ such that $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in D$. Since p and q have compact support, there exists $\bar{n} \in \mathbb{N}$ such that $[-\bar{n}e, \bar{n}e]$ contains both supports. Consider $v \in \mathcal{W}$. Since D is d-dense in $\mathcal{W}$, there exists a sequence $\{v_l\}_{l\in\mathbb{N}} \subseteq D$ such that $v_l \xrightarrow{d} v$. It follows that v_l converges uniformly on $[-\bar{n}e, \bar{n}e]$. This implies that

$$\begin{aligned}\mathbb{E}_p(v) &= \int_{[-\bar{n}e,\bar{n}e]} v \mathrm{d}p = \lim_l \int_{[-\bar{n}e,\bar{n}e]} v_l \mathrm{d}p = \lim_l \mathbb{E}_p(v_l) \\ &\geq \lim_l \mathbb{E}_q(v_l) = \lim_l \int_{[-\bar{n}e,\bar{n}e]} v_l \mathrm{d}q = \int_{[-\bar{n}e,\bar{n}e]} v \mathrm{d}q = \mathbb{E}_q(v).\end{aligned}$$

[25] A proof is available upon request.

By (14) and (16) and since v was arbitrarily chosen, we can conclude that

$$p \succcurlyeq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in D. \tag{17}$$

Since D is countable, we can list its elements: $D = \{v_m\}_{m\in\mathbb{N}}$. Set $b_l = l + \max\{|v_l(-le)|, |v_l(le)|\}$ for all $l \in \mathbb{N}$ and $a_m = \Pi_{l=1}^m b_l \geq b_m$ for all $m \in \mathbb{N}$. Finally, define $v : \mathbb{R}^k \to \mathbb{R}$ by

$$v(x) = \sum_{m=1}^{\infty} \frac{v_m(x)}{a_m} \qquad \forall x \in \mathbb{R}^k. \tag{18}$$

We first prove that v is a well-defined continuous function. Fix $x \in \mathbb{R}^k$. It follows that there exists $\bar{m} \in \mathbb{N}$ such that $x \in [-me, me]$ for all $m \geq \bar{m}$. Since each v_m is increasing, we have that $|v_m(x)| \leq \max\{|v_m(-me)|, |v_m(me)|\} \leq b_m \leq a_m$ for all $m \geq \bar{m}$. Since $a_m \geq m!$ for all $m \in \mathbb{N}$, it follows that

$$\frac{|v_m(x)|}{a_m} = \frac{|v_m(x)|}{b_m a_{m-1}} \leq \frac{1}{a_{m-1}} \leq \frac{1}{(m-1)!} \qquad \forall m \geq \bar{m} + 1.$$

This implies that the right-hand side of (18) converges. Since x was arbitrarily chosen, v is well-defined. Next, consider $n \in \mathbb{N}$. From the same argument above, we have that

$$\frac{|v_m(x)|}{a_m} \leq \frac{1}{(m-1)!} \quad \forall x \in [-ne, ne], \forall m \geq n + 1.$$

By Weierstrass' M-test and since $\{v_m/a_m\}_{m\in\mathbb{N}}$ is a sequence of continuous functions, we can conclude that $v = \sum_{m=1}^{\infty} \frac{v_m}{a_m}$ converges uniformly on $[-ne, ne]$, yielding that v is continuous on $[-ne, ne]$. Since n was arbitrarily chosen, it follows that v is continuous.

Finally, assume that $p \succ^* q$ (resp. $p \sim^* q$). By (17), we have that $\mathbb{E}_p(v_m) \geq \mathbb{E}_q(v_m)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_p(v_{\hat{m}}) > \mathbb{E}_q(v_{\hat{m}})$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_p(v_m) = \mathbb{E}_q(v_m)$ for all $m \in \mathbb{N}$). In particular, we have that $\mathbb{E}_p(v_m/a_m) \geq \mathbb{E}_q(v_m/a_m)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_p(v_{\hat{m}}/a_{\hat{m}}) > \mathbb{E}_q(v_{\hat{m}}/a_{\hat{m}})$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_p(v_m/a_m) = \mathbb{E}_q(v_m/a_m)$ for all $m \in \mathbb{N}$). Since $\sum_{m=1}^{\infty} \frac{v_m}{a_m}$ converges uniformly on compacta and the supports of p and q are compact, we can conclude that

$$\mathbb{E}_p(v) - \mathbb{E}_q(v) = \mathbb{E}_p\left(\sum_{m=1}^{\infty} \frac{v_m}{a_m}\right) - \mathbb{E}_q\left(\sum_{m=1}^{\infty} \frac{v_m}{a_m}\right) = \lim_l \sum_{m=1}^{l} \mathbb{E}_p\left(\frac{v_m}{a_m}\right) - \lim_l \sum_{m=1}^{l} \mathbb{E}_q\left(\frac{v_m}{a_m}\right)$$
$$= \lim_l \left[\sum_{m=1}^{l} \left(\mathbb{E}_p\left(\frac{v_m}{a_m}\right) - \mathbb{E}_q\left(\frac{v_m}{a_m}\right)\right)\right].$$

This implies that if $p \succ^* q$ (resp. $p \sim^* q$), then $\mathbb{E}_p(v) > \mathbb{E}_q(v)$ (resp. $\mathbb{E}_p(v) = \mathbb{E}_q(v)$), proving that v is an Aumann utility. In particular, by (15), v is strictly increasing. ■

Consider a binary relation $\succcurlyeq^*$ on Δ. Define $\mathcal{W}_{\max}(\succcurlyeq^*)$ as the set of all strictly increasing functions $v \in C(\mathbb{R}^k)$ such that $v(0) = 0$ and $p \succcurlyeq^* q$ implies $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$. We say that a set $\mathcal{W}$ in $C(\mathbb{R}^k)$ has full image if and only if

$$\forall x, y \in \mathbb{R}^k, \exists m \in \mathbb{R}_+ \text{ s.t. } v(y + me_1) \geq v(x) \geq v(y - me_1) \qquad \forall v \in \mathcal{W}.$$

Proposition 8. *Let $\succcurlyeq^*$ be a binary relation on Δ represented as in (14). If $\succcurlyeq^*$ satisfies (12) and (13), then $\mathcal{W}_{\max}(\succcurlyeq^*)$ is a nonempty convex set with full image that satisfies (14).*

Proof. Consider $v_1, v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$ and $\lambda \in (0,1)$. Since both functions are strictly increasing and continuous and such that $v_1(0) = 0 = v_2(0)$, it follows that $\lambda v_1 + (1-\lambda) v_2$ is strictly increasing, continuous, and takes value 0 in 0. Since $v_1, v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$, if $p \succcurlyeq^* q$, then $\mathbb{E}_p(v_1) \geq \mathbb{E}_q(v_1)$ and $\mathbb{E}_p(v_2) \geq \mathbb{E}_q(v_2)$. This implies that

$$\mathbb{E}_p(\lambda v_1 + (1-\lambda) v_2) = \lambda \mathbb{E}_p(v_1) + (1-\lambda) \mathbb{E}_p(v_2)$$
$$\geq \lambda \mathbb{E}_q(v_1) + (1-\lambda) \mathbb{E}_q(v_2) = \mathbb{E}_q(\lambda v_1 + (1-\lambda) v_2),$$

proving that $\lambda v_1 + (1-\lambda) v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$ and, in particular, $\mathcal{W}_{\max}(\succcurlyeq^*)$ is convex. By Proposition 7, there exists a strictly increasing $\hat{v} \in C(\mathbb{R}^k)$ such that

$$p \succ^* q \Longrightarrow \mathbb{E}_p(\hat{v}) > \mathbb{E}_q(\hat{v}) \text{ and } p \sim^* q \Longrightarrow \mathbb{E}_p(\hat{v}) = \mathbb{E}_q(\hat{v}).$$

Without loss of generality, we can assume that $\hat{v}(0) = 0$ (given $\hat{v}$, set $v = \hat{v} - \hat{v}(0)$) and, in particular, we have that $\hat{v} \in \mathcal{W}_{\max}(\succcurlyeq^*)$, proving that $\mathcal{W}_{\max}(\succcurlyeq^*)$ is nonempty. Since $\succcurlyeq^*$ satisfies (13), it follows that $\mathcal{W}_{\max}(\succcurlyeq^*)$ has full image. Since $\succcurlyeq^*$ satisfies (12), v is increasing for all $v \in \mathcal{W}$. This implies that for each $v \in \mathcal{W}$ and for each $n \in \mathbb{N}$ the function

$v_n = \left(1-\frac{1}{n}\right)v + \frac{1}{n}\hat{v} - \left[\left(1-\frac{1}{n}\right)v(0) + \frac{1}{n}\hat{v}(0)\right] \in \mathcal{W}_{\max}(\succcurlyeq^*)$. By definition, if $p \succcurlyeq^* q$, then $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}_{\max}(\succcurlyeq^*)$. Vice-versa, we have that

$$\begin{aligned}
&\mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq^*) \\
&\qquad \Longrightarrow \mathbb{E}_p\left(\left(1-\frac{1}{n}\right)v + \frac{1}{n}\hat{v}\right) \geq \mathbb{E}_q\left(\left(1-\frac{1}{n}\right)v + \frac{1}{n}\hat{v}\right) \quad \forall v \in \mathcal{W}, \forall n \in \mathbb{N} \\
&\qquad\qquad \Longrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W} \implies p \succcurlyeq^* q,
\end{aligned}$$

proving that (14) holds with $\mathcal{W}_{\max}(\succcurlyeq^*)$ in place of $\mathcal{W}$. ■

We conclude by discussing Multi-Expected Utility representations which feature odd sets. To do this, we make two simple observations. First, recall the map $\sigma : \Delta \to \Delta$, which swaps gains with losses, defined by

$$\sigma(p)(B) = p(-B) \text{ for all Borel subsets } B \text{ of } \mathbb{R}^k \text{ and for all } p \in \Delta.$$

It is immediate to see that σ is affine and $\sigma(\sigma(p)) = p$ for all $p \in \Delta$. Second, by the Change of Variable Theorem (see, e.g., Aliprantis and Border 2006, Theorem 13.46), we have that

$$\mathbb{E}_{\sigma(r)}(v) = \int_{\mathbb{R}^k} v \mathrm{d}\sigma(r) = -\int_{\mathbb{R}^k} \bar{v} \mathrm{d}r = -\mathbb{E}_r(\bar{v}) \quad \forall r \in \Delta, \forall v \in C\left(\mathbb{R}^k\right) \tag{19}$$

where $\bar{v} : \mathbb{R}^k \to \mathbb{R}$ is defined by $\bar{v}(x) = -v(-x)$ for all $x \in \mathbb{R}^k$ and for all $v \in C\left(\mathbb{R}^k\right)$.

Proposition 9. *Let $\succcurlyeq^*$ be a binary relation on Δ represented as in (14) which satisfies (12) and (13). The following statements are equivalent:*

(i) For each $p, q \in \Delta$

$$p \succcurlyeq^* q \iff \sigma(q) \succcurlyeq^* \sigma(p).$$

(ii) For each $p, q \in \Delta$

$$p \succcurlyeq^* q \implies \sigma(q) \succcurlyeq^* \sigma(p).$$

(iii) $\mathcal{W}_{\max}(\succcurlyeq^*)$ is odd.

Moreover, if $\mathcal{W}$ in (14) is odd, then (i) and (ii) hold.

For the last part of the statement, that is proving that if $\mathcal{W}$ is odd, then (i) and (ii) hold, we can dispense with the assumption that $\succcurlyeq^*$ satisfies (12) and (13). The proof will clarify.

Proof. By Proposition 8, we have that

$$p \succcurlyeq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq^*).$$

In other words, for the first part of the statement, we can replace $\mathcal{W}$ in (14) with $\mathcal{W}_{\max}(\succcurlyeq^*)$.

(i) implies (ii). It is obvious.

(ii) implies (iii). Fix $v \in \mathcal{W}_{\max}(\succcurlyeq^*)$. By definition of $\bar{v}$ and since each v in $\mathcal{W}_{\max}(\succcurlyeq^*)$ is strictly increasing, continuous, and such that $v(0) = 0$, we have that $\bar{v}$ is strictly increasing, continuous, and such that $\bar{v}(0) = 0$. By assumption and (19), we have that

$$p \succcurlyeq^* q \implies \sigma(q) \succcurlyeq^* \sigma(p) \implies \mathbb{E}_{\sigma(q)}(v) \geq \mathbb{E}_{\sigma(p)}(v) \implies -\mathbb{E}_q(\bar{v}) \geq -\mathbb{E}_p(\bar{v}) \implies \mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v}).$$

By definition of $\mathcal{W}_{\max}(\succcurlyeq^*)$, we can conclude that $\bar{v} \in \mathcal{W}_{\max}(\succcurlyeq^*)$, proving that $\mathcal{W}_{\max}(\succcurlyeq^*)$ is odd.

(iii) implies (i). By (19) and since $\mathcal{W}$ is odd and represents $\succcurlyeq^*$, we have that

$$\begin{aligned} p \succcurlyeq^* q &\iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W} \iff \mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v}) \quad \forall v \in \mathcal{W} \\ &\iff \mathbb{E}_{\sigma(q)}(v) \geq \mathbb{E}_{\sigma(p)}(v) \quad \forall v \in \mathcal{W} \iff \sigma(q) \succcurlyeq^* \sigma(p), \end{aligned}$$

proving the implication (since $\mathcal{W}_{\max}(\succcurlyeq^*)$ represents $\succcurlyeq^*$) and also the second part of the statement. ■

Representing $\succcurlyeq'$

We can finally provide a Multi-Expected Utility representation for $\succcurlyeq'$.

Proposition 10. *If $\succcurlyeq$ satisfies Weak Order, Continuity, Monotonicity, and Monetary equivalent, then*

$$p \succcurlyeq' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq').$$

Moreover, $\mathcal{W}_{\max}(\succcurlyeq')$ is a nonempty convex set with full image.

Proof. By the same techniques of Cerreia-Vioglio (2009, Proposition 22) (see also Cerreia-Vioglio et al. 2017, Lemma 1 and Footnote 10), $\succcurlyeq'$ is a preorder that satisfies Sequential Continuity and Independence.[26] By Evren (2008, Theorem 2), there exists a set $\mathcal{W} \subseteq$

[26] That is, for each two generalized sequences $\{p_\alpha\}_{\alpha \in A}$ and $\{q_\alpha\}_{\alpha \in A}$ in Δ

$$p_\alpha \succcurlyeq' q_\alpha \quad \forall \alpha \in A, \, p_\alpha \to p, \text{ and } q_\alpha \to q \implies p \succcurlyeq' q.$$

$C(\mathbb{R}^k)$ such that $p \succcurlyeq' q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$. By Lemma 1 and since $\succcurlyeq$ is a Weak Order which satisfies Monotonicity and Monetary equivalent, we have that $\succcurlyeq'$ satisfies (12) and (13). By Proposition 8 and considering $\succcurlyeq'$ in place of $\succcurlyeq^*$, $\mathcal{W}$ can be chosen to be $\mathcal{W}_{\max}(\succcurlyeq')$, proving the statement. ■

Missing Proofs

In this section, we prove Proposition 4. We begin by showing that if $\succcurlyeq$ admits a finite essential Cautious Utility representation, then it is canonical. This fact will be key in proving the aforementioned proposition.

Lemma 2. *If $\succcurlyeq$ admits a finite essential Cautious Utility representation, then it is canonical.*

Proof. Define $\succcurlyeq^*$ to be such that $p \succcurlyeq^* q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$ where $\mathcal{W}$ is a finite essential Cautious Utility representation of $\succcurlyeq$. Since $\mathcal{W}$ is finite, we have that the smallest convex cone containing $\mathcal{W}$, denoted by cone$(\mathcal{W})$, is closed with respect to the $\sigma(C(\mathbb{R}^k), \Delta)$-topology and so is the set cone$(\mathcal{W}) + \{\theta 1_{\mathbb{R}^k}\}_{\theta \in \mathbb{R}}$. By definition of $\mathcal{W}_{\max}(\succcurlyeq^*)$, it follows that cone$(\mathcal{W}) \setminus \{0\} \subseteq \mathcal{W}_{\max}(\succcurlyeq^*)$. By Proposition 8, Remark 3, and (Evren, 2008, Theorem 5) and since $\mathcal{W}$ is a Cautious Utility representation, we have that (where the closure is in the $\sigma(C(\mathbb{R}^k), \Delta)$-topology)

$$\begin{aligned} \operatorname{cone}(\mathcal{W}) + \{\theta 1_{\mathbb{R}^k}\}_{\theta \in \mathbb{R}} &= \operatorname{cl}\left(\operatorname{cone}(\mathcal{W}_{\max}(\succcurlyeq^*)) + \{\theta 1_{\mathbb{R}^k}\}_{\theta \in \mathbb{R}}\right) \\ &\supseteq \operatorname{cl}\left(\mathcal{W}_{\max}(\succcurlyeq^*) + \{\theta 1_{\mathbb{R}^k}\}_{\theta \in \mathbb{R}}\right) \\ &\supseteq \mathcal{W}_{\max}(\succcurlyeq^*) + \{\theta 1_{\mathbb{R}^k}\}_{\theta \in \mathbb{R}}, \end{aligned}$$

yielding that cone$(\mathcal{W}) \setminus \{0\} \supseteq \mathcal{W}_{\max}(\succcurlyeq^*)$ and, in particular, cone$(\mathcal{W}) \setminus \{0\} = \mathcal{W}_{\max}(\succcurlyeq^*)$. Since the functional $v \mapsto c(p, v)$ is quasiconcave over cone$(\mathcal{W}) \setminus \{0\}$ for all $p \in \Delta$, it is immediate to see that

$$V(p) = \min_{v \in \mathcal{W}} c(p, v) = \min_{v \in \operatorname{cone}(\mathcal{W}) \setminus \{0\}} c(p, v) \quad \forall p \in \Delta.$$

By Remark 3 and since $\mathcal{W} = \{v_i\}_{i=1}^n$ is a finite Cautious Utility representation, we have that $\succcurlyeq$ satisfies Axioms 1- 5. By Theorem 1 and its proof, $\mathcal{W}_{\max}(\succcurlyeq')$ is a canonical Cautious

Utility representation for $\succcurlyeq$. In particular, we have that

$$V(p) = \min_{v \in \mathcal{W}} c(p, v) = \min_{v \in \mathrm{cone}(\mathcal{W}) \setminus \{0\}} c(p, v) = \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(p, v) \quad \forall p \in \Delta.$$

Since $\succcurlyeq'$ is the largest subrelation of $\succcurlyeq$ that satisfies the Independence axiom and $p \succcurlyeq^* q$ implies $p \succcurlyeq q$, we have that $\succcurlyeq^*$ is a subrelation of $\succcurlyeq'$ and $\mathcal{W}_{\max}(\succcurlyeq') \subseteq \mathcal{W}_{\max}(\succcurlyeq^*) = \mathrm{cone}(\mathcal{W}) \setminus \{0\}$. By contradiction, assume that $\mathcal{W}_{\max}(\succcurlyeq') \neq \mathrm{cone}(\mathcal{W}) \setminus \{0\}$. Since $\mathcal{W}_{\max}(\succcurlyeq')$ is a convex set closed with respect to strictly positive scalar multiplications, this implies that $\mathcal{W} \not\subseteq \mathcal{W}_{\max}(\succcurlyeq')$. If $\mathcal{W}$ is a singleton, then $\succcurlyeq$ is Expected Utility and, in particular, $\succcurlyeq'$ is complete and coincides with $\succcurlyeq$. This implies that $\mathcal{W} = \{v_1\}$ and $\mathcal{W}_{\max}(\succcurlyeq') = \{\lambda v_1\}_{\lambda > 0} = \mathrm{cone}(\mathcal{W}) \setminus \{0\}$, a contradiction. Assume $\mathcal{W}$ is not a singleton. Consider $\breve{v} \in \mathcal{W} \setminus \mathcal{W}_{\max}(\succcurlyeq')$. Since $\mathcal{W}$ is essential, there exists $\bar{p} \in \Delta$ such that $\min_{v \in \mathcal{W}} c(\bar{p}, v) < \min_{v \in \mathcal{W} \setminus \{\breve{v}\}} c(\bar{p}, v)$. Since $\mathcal{W} = \{v_i\}_{i=1}^n$ and $n \geq 2$, without loss of generality, we can set $\breve{v} = v_n \notin \mathcal{W}_{\max}(\succcurlyeq')$. In particular, we have that

$$\inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v) = \min_{v \in \mathcal{W}} c(\bar{p}, v) = c(\bar{p}, v_n) < c(\bar{p}, v_i) \quad \forall i \in \{1, \ldots, n-1\}. \tag{20}$$

Consider a sequence $\{\hat{v}_m\}_{m \in \mathbb{N}} \subseteq \mathcal{W}_{\max}(\succcurlyeq')$ such that $c(\bar{p}, \hat{v}_m) \downarrow \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v)$. By construction and since $\mathcal{W}_{\max}(\succcurlyeq') \subseteq \mathrm{cone}(\mathcal{W}) \setminus \{0\}$, there exists a collection of scalars $\left\{\lambda_{m,i}\right\}_{m \in \mathbb{N}, i \in \{1, \ldots, n\}} \subseteq [0, \infty)$ such that $\hat{v}_m = \sum_{i=1}^n \lambda_{m,i} v_i$ for all $m \in \mathbb{N}$. Since $\hat{v}_m$ is strictly increasing, we have that for each $m \in \mathbb{N}$ there exists $i \in \{1, \ldots, n\}$ such that $\lambda_{m,i} > 0$. Define $\lambda_{m,\sigma} = \sum_{i=1}^n \lambda_{m,i} > 0$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ and for each $i \in \{1, \ldots, n\}$ define also $\bar{\lambda}_{m,i} = \lambda_{m,i} / \lambda_{m,\sigma}$ as well as $\tilde{v}_m = \sum_{i=1}^n \bar{\lambda}_{m,i} v_i = \hat{v}_m / \lambda_{m,\sigma}$. Since $\lambda_{m,\sigma} > 0$ for all $m \in \mathbb{N}$, it is immediate to see that $c(\bar{p}, \tilde{v}_m) = c(\bar{p}, \hat{v}_m)$ for all $m \in \mathbb{N}$ and, in particular, $c(\bar{p}, \tilde{v}_m) \downarrow \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v)$. For each $m \in \mathbb{N}$ denote by $\bar{\lambda}_m$ the $\mathbb{R}^n$ vector whose i-th component is $\bar{\lambda}_{m,i}$. Since $\left\{\bar{\lambda}_m\right\}_{m \in \mathbb{R}}$ is a sequence in the $\mathbb{R}^n$ simplex, there exists a subsequence $\left\{\bar{\lambda}_{m_l}\right\}_{l \in \mathbb{N}}$ such that $\bar{\lambda}_{m_l,i} \longrightarrow \bar{\lambda}_i \in [0, 1]$ for all $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^n \bar{\lambda}_i = 1$. It is immediate to see that $\tilde{v}_{m_l} = \sum_{i=1}^n \bar{\lambda}_{m_l,i} v_i \overset{\sigma(C(\mathbb{R}^k), \Delta)}{\longrightarrow} \sum_{i=1}^n \bar{\lambda}_i v_i = \tilde{v}$ where $\tilde{v}$ is continuous, strictly increasing, and such that $\tilde{v}(0) = 0$. Moreover, for each $p, q \in \Delta$ we have that $p \succcurlyeq' q$ implies $\mathbb{E}_p(\tilde{v}) \geq \mathbb{E}_q(\tilde{v})$, proving that $\tilde{v} \in \mathcal{W}_{\max}(\succcurlyeq')$. Note that $\bar{\lambda}_n < 1$, otherwise, we would have that $v_n = \tilde{v} \in \mathcal{W}_{\max}(\succcurlyeq')$, a contradiction. By (20) and since $\bar{\lambda}_n < 1$ and the functional $v \mapsto c(p, v)$ is explicitly quasiconcave over $\text{co}(\mathcal{W})$ for all $p \in \Delta$,[27] we have that

$$c(\bar{p}, v_n) < c(\bar{p}, \tilde{v}) = \lim_l c\left(\bar{p}, \tilde{v}_{m_l}\right) = \lim_m c(\bar{p}, \tilde{v}_m) = \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v) = c(\bar{p}, v_n),$$

a contradiction. It follows that $\mathcal{W}_{\max}(\succcurlyeq') = \text{cone}(\mathcal{W}) \setminus \{0\}$ and, in particular, $\mathcal{W}$ represents also $\succcurlyeq'$. This implies that $\mathcal{W}$ is canonical. ∎

Proof of Proposition 4. We first prove the first part of the statement assuming $\succcurlyeq$ satisfies u-CPT, and then we will move to the additive case. Since $u(0) = 0$ and u is strictly increasing and continuous, it follows that there exists $\bar{t} > 0$ such that $[-\bar{t}, \bar{t}] \subseteq \text{Im}\, u$. Let $\Delta_0([0, \bar{t}])$ be the set of finitely supported probabilities over $[0, \bar{t}]$. Consider $\tilde{p} \in \Delta_0([0, \bar{t}])$. By definition, we have that there exist two unique collections $\{t_i\}_{i=1}^n \subseteq [0, \bar{t}]$ and $\{\lambda_i\}_{i=1}^n \subseteq [0, 1]$ such that $\text{supp}\, p = \{t_i\}_{i=1}^n$, $\sum_{i=1}^n \lambda_i = 1$, and $\tilde{p} = \sum_{i=1}^n \lambda_i \delta_{t_i}$. Without loss of generality, we can assume that $t_1 < \ldots < t_n$. We define $\tilde{V} : \Delta_0([0, \bar{t}]) \to \mathbb{R}$ by

$$\tilde{V}(\tilde{p}) = \sum_{j=1}^{n-1} \left(\bar{w}^+ \left(\sum_{i=j}^{n} \lambda_i \right) - \bar{w}^+ \left(\sum_{i=j+1}^{n} \lambda_i \right) \right) v\left(t_j\right) + \bar{w}^+(\lambda_n)\, v(t_n)$$

for all $\tilde{p} \in \Delta_0([0, \bar{t}])$ where $\bar{w}^+ : [0, 1] \to [0, 1]$ is defined by $\bar{w}^+(t) = 1 - w(1 - t)$ for all $t \in [0, 1]$. We next show that for each $\tilde{p} \in \Delta_0([0, \bar{t}])$ and for each $\tilde{t} \in [0, \bar{t}]$, if $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$, then $\tilde{V}(\lambda \tilde{p} + (1 - \lambda)\delta_{\tilde{t}}) = \tilde{V}(\delta_{\tilde{t}})$ for all $\lambda \in (0, 1)$. Consider $\tilde{p} \in \Delta_0([0, \bar{t}])$ and $\tilde{t} \in [0, \bar{t}]$ such that $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$. Given $\tilde{p} \in \Delta_0([0, \bar{t}])$, since $\{t_i\}_{i=1}^n \subseteq [0, \bar{t}] \subseteq \text{Im}\, u$, there exists $\{x_i\}_{i=1}^n \subseteq \mathbb{R}^k$ such that $u(x_i) = t_i$ for all $i \in \{1, \ldots, n\}$. Consider $p = \sum_{i=1}^n \lambda_i \delta_{x_i}$. It is immediate to see that $\tilde{V}(\tilde{p}) = V(p)$. Since $\succcurlyeq$ admits a Symmetric Cautious Utility representation, there exists $c \in \mathbb{R}$ such that $p \sim \delta_{ce_1}$. This implies that $V(p) = V\left(\delta_{ce_1}\right)$ and, in particular, $u(ce_1) \in [0, \bar{t}]$. Moreover, since u and v are strictly increasing, we have that $u(ce_1) = \tilde{t} \in [0, \bar{t}]$ and $V\left(\delta_{ce_1}\right) = \tilde{V}(\delta_{\tilde{t}})$. By Remark 3 and since $\succcurlyeq$ admits a Symmetric Cautious Utility representation, we have that $\succcurlyeq$ satisfies M-NCI. This yields that $\lambda p + (1 - \lambda)\delta_{ce_1} \sim \delta_{ce_1}$ for all $\lambda \in (0, 1)$. This implies that

$$\tilde{V}(\lambda \tilde{p} + (1 - \lambda)\delta_{\tilde{t}}) = V\left(\lambda p + (1 - \lambda)\delta_{ce_1}\right) = V\left(\delta_{ce_1}\right) = \tilde{V}(\delta_{\tilde{t}}).$$

[27]Formally, see e.g. (Aliprantis and Border, 2006, p. 300), given $p \in \Delta$, for each $h \in \mathbb{N} \setminus \{1\}$, for each $\{v_l\}_{l=1}^h \subseteq \text{co}(\mathcal{W})$, and for each $\{\lambda_l\}_{l=1}^h \subseteq [0, 1]$ such that $\sum_{l=1}^h \lambda_l = 1$ and $\lambda_h < 1$

$$c(p, v_i) > c(p, v_h) \quad \forall i \in \{1, \ldots, h-1\} \implies c\left(p, \sum_{i=1}^h \lambda_i v_i\right) > c(p, v_h).$$

By Bell and Fishburn (2003, Theorem 1) applied to $\tilde{V}$, it follows that $\bar{w}^+$ is the identity and so is w^+. The same proof, performed with $[-\bar{t}, 0]$ in place of $[0, \bar{t}]$ and w^+ replaced by w^-, yields that w^- is the identity. These two facts together allow us to conclude that $p \mapsto V(p) = \text{CPT}_{v,w^+,w^-}(p_u)$ is an Expected Utility functional with utility $v \circ u : \mathbb{R}^k \to \mathbb{R}$.

We next assume that $\succcurlyeq$ admits an Additive CPT representation. As before consider $\bar{t} > 0$. Define $\Delta_0([0, \bar{t}])$ and $\tilde{V}$ as before with v replaced by u_1. For each $\tilde{p} \in \Delta_0([0, \bar{t}])$ define p in Δ to be the product measure $\tilde{p} \otimes \delta_0 ... \otimes \delta_0$. It is immediate to see that $\tilde{V}(\tilde{p}) = V(p)$ for all $\tilde{p} \in \Delta_0([0, \bar{t}])$. As before, we can show that for each $\tilde{p} \in \Delta_0([0, \bar{t}])$ and for each $\tilde{t} \in [0, \bar{t}]$, if $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$, then $\tilde{V}(\lambda\tilde{p} + (1-\lambda)\delta_{\tilde{t}}) = \tilde{V}(\delta_{\tilde{t}})$ for all $\lambda \in (0, 1)$. Consider $\tilde{p} \in \Delta_0([0, \bar{t}])$ and $\tilde{t} \in [0, \bar{t}]$ such that $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$. This implies that $V(p) = V\left(\delta_{\tilde{t}e_1}\right)$, that is, $p \sim \delta_{\tilde{t}e_1}$. By Remark 3 and since $\succcurlyeq$ admits a Symmetric Cautious Utility representation, we have that $\succcurlyeq$ satisfies M-NCI. This yields that $\lambda p + (1-\lambda)\delta_{\tilde{t}e_1} \sim \delta_{\tilde{t}e_1}$ for all $\lambda \in (0, 1)$. This implies that

$$\tilde{V}(\lambda\tilde{p} + (1-\lambda)\delta_{\tilde{t}}) = V\left(\lambda p + (1-\lambda)\delta_{\tilde{t}e_1}\right) = V\left(\delta_{\tilde{t}e_1}\right) = \tilde{V}(\delta_{\tilde{t}}).$$

By Bell and Fishburn (2003, Theorem 1) applied to $\tilde{V}$, it follows that $\bar{w}^+$ is the identity and so is w^+. The same proof, performed with $[-\bar{t}, 0]$ in place of $[0, \bar{t}]$ and w^+ replaced by w^-, yields that w^- is the identity. This implies that $\succcurlyeq$ admits an Expected Utility representation with utility $u : \mathbb{R}^k \to \mathbb{R}$ defined by $u(x) = \sum_{i=1}^k u_i(x_i)$ for all $x \in \mathbb{R}^k$.

As for the second part of the statement, by Lemma 2 and since $\mathcal{W}$ is a finite essential Cautious Utility representation, we have that $\mathcal{W}$ is a canonical representation, that is, $\mathcal{W} = \{v_i\}_{i=1}^n$ represents also $\succcurlyeq'$. Since $\succcurlyeq$ is Expected Utility with utility $v \circ u$ (where in the additive case v is the identity and u is additively separable), we have that $\succcurlyeq'$ coincides with $\succcurlyeq$, yielding that for each $i \in \{1, ..., n\}$ there exists $\lambda_i > 0$ such that $v_i = \lambda_i(v \circ u)$. This implies that $c(p, v_i) = c(p, v \circ u)$ for all $p \in \Delta$ and for all $i \in \{1, ..., n\}$. Since $\mathcal{W}$ is essential, this implies that $\mathcal{W}$ is a singleton. Since $\mathcal{W} = \{v_1\}$ and $\mathcal{W}$ is odd, this implies that v_1 is odd and, in particular, $\succcurlyeq$ is loss neutral for risk and exhibits no endowment effect. ■

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