ONLINE APPENDIX

A Evidence on Comparative and Ordinal Reasoning

A core component of our model is that, when people are uncertain how to judge or value objects, they rely on comparators as a source of information. This section reviews evidence that suggests that people indeed often rely on comparators (in a partially ordinal way) when they are uncertain what the right decision is.

The broad idea that decision making is often comparative in nature is one of the central and earliest insights of behavioral economics, as evidenced in prospect theory. A classic example is that warm water feels warmer after we stick our hand into cold rather than hot water. However, the comparative nature of cognition and decision making is not limited to reference-dependent preferences. Kahneman and Tversky (1979) already note that "the perceptual apparatus is attuned to the evaluation of changes or differences rather than to the evaluation of absolute magnitudes". Indeed, cognitive psychology has long emphasized that people are more adept at judging differences between stimuli than estimating absolute magnitudes.

Decades of psychophysical research suggest that performance on absolute judgment tasks – evaluating the magnitude of a stimulus – is substantially poorer than on tasks that require comparisons between two stimuli – evaluating which one is brighter, longer, heavier etc. This supports the view that human perception is optimized for detecting differences rather than encoding absolute magnitudes (Laming, 2009, 1984; Braida and Durlach, 1972; Garner, 1953). Empirically, the precision of the ordering of two stimuli implied by cardinal estimates is about two orders of magnitude lower than the precision of direct comparisons between the same stimuli (Laming, 1997). Moreover, responses in the classic psychological absolute judgment tasks (magnitude estimation and absolute identification) are more strongly shaped by contextual factors than in the comparative task of ordinal stimulus discrimination (Luce and Green, 1974; Parducci, 1965; Laming, 1997), suggesting that comparisons are more stable.

Much research suggests that such comparative thinking is partly ordinal in nature. It is generally easier to assess whether one stimulus is greater than, less than, or equal to another ones than it is to make cardinal judgments. The importance of ordinal judgments has been most prominent in perceptual tasks but also heavily influenced work on decision-making. For example, it is intuitively easier to realize that six apples are better than four apples, compared to assessing how much better they are. One clear illustration of this in the perceptual domain is Hollands and Dyre (2000) who show that in magnitude estimation tasks with an exogenously-provided reference point of known value, subjects correctly recognize whether a stimulus is above or below the reference point but are insufficiently sensitive to how much, the two hallmark characteristics of partiall ordinal reasoning.

Importantly, much evidence suggests that people partly implicitly rely on ordinal comparisons even when they produce cardinal estimates. A classic reference is Braida and Durlach (1972), who showed that magnitude estimation (assigning a number proportional to intensity) and absolute identification (judging which stimulus is larger) yield essentially the same pattern of responses. Stewart et al. (2005) review this and related empirical observations, and propose a psychological model of absolute identification through relative judgment. Relatedly, the influential line of work on decision-by-sampling essentially asserts that people make decisions by ordinally comparing options with comparators retrieved from memory, and counting the number of advantages (Stewart et al., 2006, and references therein).

More recently, some contributions in economics suggested that relying on comparative thinking may partly reflect a simplification strategy and / or incomplete information about how to solve a problem. In studying belief updating, Augenblick et al. (2025) model a DM who knows in which direction to update (ordinally) but not by how much. Graeber et al. (2025) and Graeber ⓡ Enke (2026) show empirically that higher processing difficulty or uncertainty lead to stronger reliance on comparison points, and that such comparative thinking appears to be partly ordinal in nature.

In summary, when people exhibit uncertainty about how to estimate the value or magnitude of a stimulus, they partially rely on comparisons, be they explicitly provided or retrieved from memory. Given its low cost and simplicity, these comparisons are partly ordinal in nature. Unlike some of the more extreme hypotheses and models in psychology, we do not posit that people only rely on comparative or ordinal information. Instead, our model features a standard absolute, cardinal signal, augmented with a comparative, ordinal one.[32]

B Additional Results

We now present several additional formal results. All proofs for this material appear in Appendix E.

[32]A notable caveat is that while humans generally perform better at, and tend to rely on, ordinal judgments, these judgments are not perfect, either. In particular, they break down when stimulus differences fall below the so-called "just noticeable difference" threshold, which we abstract away from in this paper.

B.1 Belief Updating with Correlated Signals

It is useful to prove a more general version of Proposition 1 that allows for correlation between the two varaiables.

Proposition 10. *Let* (X, Y) *be jointly normally distributed with*

$$\mathbb{E}[X] = \mu_X, \quad \mathbb{E}[Y] = \mu_Y, \quad \mathrm{Var}(X) = \sigma_X^2, \quad \mathrm{Var}(Y) = \sigma_Y^2, \quad \mathrm{Cov}(X, Y) = \sigma_{XY}.$$

Let $E \sim \mathcal{N}(0, \sigma_E^2)$ *be independent of* (X, Y)*, and define the latent index*

$$S \;:=\; X - Y + E.$$

Assume $\sigma_S^2 := \mathrm{Var}(S) > 0$*, and suppose the observer only learns that* $S > 0$*. Then*

$$\mathbb{E}[X \mid S > 0] = \mu_X + \frac{\sigma_X^2 - \sigma_{XY}}{\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY} + \sigma_E^2}} \cdot \frac{\phi\left(\dfrac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY} + \sigma_E^2}}\right)}{\Phi\left(\dfrac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY} + \sigma_E^2}}\right)}.$$

Here $\phi(u) = (2\pi)^{-1/2}\exp(-u^2/2)$ *and* $\Phi(u) = \int_{-\infty}^{u} \phi(z)\, dz$ *denote the standard normal pdf and cdf, respectively.*

If the observer instead learns $S < 0$ *then*

$$\mathbb{E}[X \mid S < 0] = \mu_X - \frac{\sigma_X^2 - \sigma_{XY}}{\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY} + \sigma_E^2}} \cdot \frac{\phi\left(\dfrac{\mu_Y - \mu_X}{\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY} + \sigma_E^2}}\right)}{\Phi\left(\dfrac{\mu_Y - \mu X}{\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY} + \sigma_E^2}}\right)}.$$

B.2 Cardinal Comparative Signals

A seemingly strong assumption of our model is that the comparative signal is only ordinal in nature—the DM may observe that u_x is better than u_r but not by how much. Yet in some settings, the DM may also have a sense of *how much* better or worse u_x is. We now provide two results. First, show that the setting we study in the paper—a cardinal signal about u_x and an ordinal signal about $u_x - u_r$—is, in fact, informationally *equivalent* to a setting in which the DM also receives a cardinal signal about $u_x - u_r$ simply by adapting the parameters, under mild assumptions on the correlation matrix. Second, we provide an extension of our basic model that allows for more general cardinal comparative signals, showing how these are easy to incorporate.

B.2.1 An Equivalence Result

In the *Original Setting* studied in the main body of the paper, parametrized by $(\sigma^2_{\varepsilon_x}, \sigma^2_{\varepsilon_r}, \sigma^2_v)$, the DM holds independent priors $u_x \sim \mathcal{N}(\tilde{u}_x, \sigma^2_x)$, $u_r \sim \mathcal{N}(\tilde{u}_r, \sigma^2_r)$ and observes cardinal signals $s_x = u_x + \varepsilon_x$ and $s_r = u_r + \varepsilon_r$, with $\varepsilon_x \sim \mathcal{N}(0, \sigma^2_{\varepsilon_x})$ and $\varepsilon_r \sim \mathcal{N}(0, \sigma^2_{\varepsilon_r})$, and an ordinal signal $o = \text{sign}(u_x - u_r - v)$ where $v \sim \mathcal{N}(0, \sigma^2_v)$.[33] All random variables are independent.

Consider instead the following *Alternative Setting*, parametrized by $(\sigma^2_{e_x}, \sigma^2_{e_d}, \sigma^2_{e_r}, \sigma^2_{v'})$, the DM holds the same priors and receives cardinal signals $s'_x = u_x + e_x$ and $s'_r = u_r + e_r$, an ordinal signal $o' = \text{sign}(u_x - u_r - v')$ where $v' \sim \mathcal{N}(0, \sigma^2_{v'})$ (independent of everything else), and a *cardinal comparative signal* $s_d = u_x - u_r + e_d$. Assume that the error vector (e_x, e_d, e_r) is independent of (u_x, u_r) and distributed as $\mathcal{N}(\mathbf{0}, \Sigma)$, where

$$\Sigma = \begin{pmatrix} \sigma^2_{e_x} & \sigma^2_{e_x} & 0 \\ \sigma^2_{e_x} & \sigma^2_{e_d} & \sigma^2_{e_r} \\ 0 & \sigma^2_{e_r} & \sigma^2_{e_r} \end{pmatrix},$$

with $\sigma^2_{e_d} > \sigma^2_{e_x} + \sigma^2_{e_r}$. This covariance structure implies that we can write $e_d = e_x + e_r + t$, where $t \sim \mathcal{N}(0, \sigma^2_{e_d} - \sigma^2_{e_x} - \sigma^2_{e_r})$ is independent of e_x and e_r: intuitively, the comparative signal s_d is subject to the noise from the individual components plus additional independent noise.

Despite their different formulations, the Original and Alternative settings are informationally equivalent: for every Alternative Setting, there exists a corresponding Original Setting, and vice-versa, that generates the same distribution of posterior beliefs about (u_x, u_r). This equivalence is formalized in the following Observation (proved below for completeness).

Observation 3 (Equivalence of Settings)**.** *The Alternative and Original Settings are informationally equivalent: they generate the same distribution of posterior beliefs over (u_x, u_r) up to a reparametrization. Specifically:*

1. ***From Alternative to Original:*** *Any Alternative Setting $(\sigma^2_{e_x}, \sigma^2_{e_d}, \sigma^2_{e_r}, \sigma^2_{v'})$ generates the same*

[33] The DM's knowledge of u_r can be interpreted as arising from a prior together with a cardinal signal about u_r, left implicit above.

distribution of posterior beliefs as the Original Setting $(\sigma^2_{\varepsilon_x}, \sigma^2_{\varepsilon_r}, \sigma^2_v)$ *such that*

$$\sigma^2_{\varepsilon_x} = \sigma^2_{e_x}, \qquad \frac{1}{\sigma^2_{\varepsilon_r}} = \frac{1}{\sigma^2_{e_r}} + \frac{4}{\sigma^2_{e_d} - \sigma^2_{e_x} - \sigma^2_{e_r}}, \qquad \sigma^2_v = \sigma^2_{v'}.$$

2. ***From Original to Alternative:*** *Conversely, any Original Setting* $(\sigma^2_{\varepsilon_x}, \sigma^2_{\varepsilon_r}, \sigma^2_v)$ *generates the same distribution of posterior beliefs as any Alternative Setting such that* $\sigma^2_{e_x} = \sigma^2_{\varepsilon_x}$, $\sigma^2_{v'} = \sigma^2_v$, *any* $\sigma^2_{e_r} > \sigma^2_{\varepsilon_r}$, *and*

$$\sigma^2_{e_d} = \sigma^2_{\varepsilon_x} + \sigma^2_{e_r} + 4\left(\frac{1}{\sigma^2_{\varepsilon_r}} - \frac{1}{\sigma^2_{e_r}}\right)^{-1}.$$

B.2.2 A model with an unrestricted comparative cardinal signal

Proposition 11. *Consider the setup of Proposition 1, but additionally the DM receives a signal* $d \sim N(x - r, \sigma^2_\varepsilon)$. *Then*

$$E(u_x|o, s, d) = \hat{u}^{s,d}_x + \frac{\bar{\sigma}^2_x - \bar{\sigma}^2_{xr}}{\sqrt{\bar{\sigma}^2_x + \bar{\sigma}^2_r - \bar{\sigma}^2_{xr} + \sigma^2_o}}\psi\left(\frac{\hat{u}^{s,d}_x - \hat{u}^{s,d}_r}{\sqrt{\bar{\sigma}^2_x + \bar{\sigma}^2_r - \bar{\sigma}^2_{xr} + \sigma^2_o}}\right)$$

where

$$\begin{aligned}
\hat{u}^{s,d}_x &= \left(1 - \frac{\sigma^2_x}{\sigma^2_x + \sigma^2_r + \sigma^2_\varepsilon}\right)\hat{\mu}^s_x + \frac{\sigma^2_x}{\sigma^2_x + \sigma^2_r + \sigma^2_\varepsilon}(d + \mu_r) \\
\hat{u}^{s,d}_r &= \left(1 - \frac{\sigma^2_r}{\sigma^2_x + \sigma^2_r + \sigma^2_\varepsilon}\right)\mu_r - \frac{\sigma^2_x}{\sigma^2_x + \sigma^2_r + \sigma^2_\varepsilon}(d - \hat{\mu}^s_x) \\
\bar{\sigma}^2_x &= \sigma^2_x\left(1 - \frac{\bar{\sigma}^2_x}{\sigma^2_x + \sigma^2_r + \sigma^2_\varepsilon}\right) \\
\bar{\sigma}^2_r &= \sigma^2_r\left(1 - \frac{\bar{\sigma}^2_r}{\sigma^2_x + \sigma^2_r + \sigma^2_\varepsilon}\right) \\
\bar{\sigma}^2_{xr} &= \frac{-\sigma^2_x\sigma^2_r}{\sigma^2_x + \sigma^2_r + \sigma^2_\varepsilon}
\end{aligned}$$

B.3 Posterior Variances After Ordinal Signals and Average Posterior Means

We now present two additional results: the characterization of the posterior variances after the ordinal signal as well as the characterization of the *average* posterior mean that the agent will have—averaging over the realizations of the (noisy) cardinal and ordinal signals.

Proposition 12 (Posterior Variances)**.** *The following is true:*

$$var(u_x|s,o) = \sigma_x^2 - \frac{\sigma_x^4}{\sigma_x^2+\sigma_r^2+\sigma_o^2}\left(t\psi(t,o)+\psi(t,o)^2\right)$$

Proposition 13 (Average Posterior Means)**.** *For all $s \in \mathbb{R}$, let $z_s := \frac{\tilde{u}_x^s - \tilde{u}_r}{\sqrt{\tilde{\sigma}_x^2+\sigma_r^2+\sigma_o^2}}$. Then,*

$$\mathbb{E}[\tilde{u}_x^{s,o}] = \lambda u_x + (1-\lambda)\tilde{u}_x + \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}\,\mathbb{E}_s\left[\left(\Phi\left(\frac{u_x-u_r}{\sigma_o}\right)-\Phi(z_s)\right)\frac{\phi(z_s)}{\Phi(z_s)\big(1-\Phi(z_s)\big)}\right]. \tag{15}$$

B.4 Comparative Statics of Reference Dependence

We analyze comparative statics both for specific signal realizations and in expectation. Let $B(+,s)$ denote the average effect of the ordinal signal + given cardinal signal s, and $C(u_x,u_r)$ the average effect of the ordinal signal given true values of u_x and u_r aggregating across possible ordinal and cardinal signals. That is

$$B(+,s) := \mathbb{E}[u_x|+,s] - \tilde{u}_x^s \quad \text{and} \quad C(u_x,u_r) := \mathbb{E}_{o,s}\left[\mathbb{E}[u_x|o,s]|u_x,u_r\right] - \mathbb{E}_s\left[\mathbb{E}[u_x|s]|u_x\right].$$

From Proposition 1, let $z_s := \frac{\tilde{u}_x^s-\tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}$, then

$$B(s,+) = \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}\frac{\phi(z_s)}{\Phi(z_s)}, \qquad B(s,-) = -\frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}\frac{\phi(-z_s)}{\Phi(-z_s)}.$$

The following proposition describes its comparative statics.

Proposition 14 (Boost from the ordinal signal)**.** *Suppose $\sigma_x^2 > 0$. Fix a realized cardinal signal s. Then, $B(s,+) > 0$ and $B(s,-) < 0$. Moreover:*

(a) Holding $\tilde{u}_x^s$ fixed, $B(s,+)$ is strictly increasing in σ_x^2.

(b) For each $v \in \{\sigma_r^2, \sigma_o^2\}$,

$$\frac{\partial B(s,+)}{\partial v} < 0 \quad \text{if } z_s \le 0.$$

If $z_s > 0$, there exists a unique cutoff $z^* \simeq 0.84$ such that

$$\frac{\partial B(s,+)}{\partial v} < 0 \quad \text{for } 0 < z_s < z^*, \qquad \text{and} \qquad \frac{\partial B(s,+)}{\partial v} > 0 \quad \text{for } z_s > z^*.$$

(c) $B(s,+)$ is strictly decreasing in $\tilde{u}_x^s - \tilde{u}_r$.

(d) *(Primitive variances.) The effect of* $(\sigma_p^2, \sigma_\epsilon^2)$ *on* $B(s,+)$ *decomposes as*

$$\frac{\partial B(s,+)}{\partial \sigma_p^2} = B_{\sigma_x^2}\frac{\partial \sigma_x^2}{\partial \sigma_p^2} + B_{\tilde{u}_x^s}\frac{\partial \tilde{u}_x^s}{\partial \sigma_p^2}, \qquad \frac{\partial B(s,+)}{\partial \sigma_\epsilon^2} = B_{\sigma_x^2}\frac{\partial \sigma_x^2}{\partial \sigma_\epsilon^2} + B_{\tilde{u}_x^s}\frac{\partial \tilde{u}_x^s}{\partial \sigma_\epsilon^2},$$

where $B_{\sigma_x^2} > 0$ *(by part (a)) and* $B_{\tilde{u}_x^s} < 0$ *(by part (c)). In particular,*

$$\frac{\partial B(s,+)}{\partial \sigma_p^2} > 0 \ \text{ whenever } s \le \tilde{u}_x, \qquad \text{and} \qquad \frac{\partial B(s,+)}{\partial \sigma_\epsilon^2} > 0 \ \text{ whenever } s \ge \tilde{u}_x.$$

Parts (a) and (c) show how the boost is strictly *increasing* in σ_x^2 and *decreasing* in $\tilde{u}_x^s - \tilde{u}_r$. Both are intuitive: The more uncertain he is about u_x, captured by σ_x^2, the more the DM will rely on the ordinal information. Similarly, the larger $\tilde{u}_x^s - \tilde{u}_r$, the smaller the surprise of a + ordinal signal, thus the smaller its effect.

More subtle are the effects of σ_r^2 and σ_o^2, discussed in Part (b). Intuitively, one may expect the two noises to have a *negative* effect on the boost—at the end of the day, both make the ordinal information less precise. Indeed, this is the case when $\tilde{u}_x^s \le \tilde{u}_r$ or when $\tilde{u}_x^s - \tilde{u}_r$ is positive but small enough. However, if $\tilde{u}_x^s - \tilde{u}_r$ is very large, an increase in σ_r^2 or σ_o^2 may *increase* the boost. To see why, suppose that $\tilde{u}_x^s - \tilde{u}_r$ is extremely large and suppose first that there is no noise in the ordinal signal. Then, a + signal is not very informative—the DM expected this to be the case already. However, crucially, it will be at least a bit more informative when σ_r^2 is higher—because even if $\tilde{u}_r$ is much below $\tilde{u}_x^s$, a high σ_r^2 increases the chance that u_r is above $\tilde{u}_r$ and therefore makes the signal more relevant. Similarly, suppose that $\tilde{u}_x^s - \tilde{u}_r$ is extremely large and $\sigma_r^2 = 0$. Again, adding a bit of noise to the ordinal signal may (paradoxically) make it *more* informative because in our model, the chances of a correct ordinal signal depend on the distance between the realized u_x and u_r; therefore, noise in the ordinal signal makes the + signal more meaningful—because, with noise, a − signal could have been possible, but it didn't realize, increasing the chances that u_x is well above u_r.

We now turn to the average boost C, for which similar comparative statics also hold.

Proposition 15 (Average boost). *Suppose* $u_x > u_r = \tilde{u}_r = \tilde{u}_x$, $\sigma_p^2 > 0$, $\sigma_\epsilon^2 > 0$, *and* $\sigma_r^2 + \sigma_o^2 > 0$. *Then,* $C(u_x, u_r) > 0$. *Moreover:*

(a) *Holding fixed the* λ, $C(u_x, u_r)$ *is strictly increasing in* σ_x^2.

(b) *For each* $v \in \{\sigma_r^2, \sigma_o^2\}$, *the effect of* v *on* $C(u_x, u_r)$ *is strictly negative when* $u_x - u_r$ *is sufficiently small, while if it is sufficiently large the derivative may become positive.*

(c) *Let* $\delta := u_x - u_r$:

(i) *If* $\sigma_o^2 = 0$*, then* $C(u_x, u_r)$ *is strictly decreasing in* δ*.*

(ii) *If* $\sigma_o^2 > 0$*, then* $C(u_x, u_r) \to 0$ *as* $\delta \downarrow 0$ *and as* $\delta \to \infty$*, and there exists* $\Delta^* > 0$ *such that* $C(u_x, u_r)$ *is increasing in* δ *on* $[0, \Delta^*]$ *and strictly in* δ *decreasing for all* $\delta > \Delta^*$*.*

(d) *The general effects of varying* σ_ϵ^2 *and* σ_p^2 *are as follows:*

$$\frac{\partial C}{\partial \sigma_p^2} = C_{\sigma_x^2} \frac{\partial \sigma_x^2}{\partial \sigma_p^2} + C_\lambda \frac{\partial \lambda}{\partial \sigma_p^2}, \qquad \frac{\partial C}{\partial \sigma_\epsilon^2} = C_{\sigma_x^2} \frac{\partial \sigma_x^2}{\partial \sigma_\epsilon^2} + C_\lambda \frac{\partial \lambda}{\partial \sigma_\epsilon^2},$$

where $C_{\sigma_x^2} > 0$ *by part (a),* $\partial\sigma_x^2/\partial\sigma_p^2 > 0$ *and* $\partial\sigma_x^2/\partial\sigma_\epsilon^2 > 0$*, while* $\partial\lambda/\partial\sigma_p^2 > 0$ *and* $\partial\lambda/\partial\sigma_\epsilon^2 < 0$*. Thus, changes in* σ_p^2 *and* σ_ϵ^2 *always raise* C *through the posterior-variance channel* σ_x^2*, but they move* λ *in opposite directions.*

Notice that the average boost C may be *negative* even if $u_x > u_r$ when $\tilde{u}_x$ is much above $\tilde{u}_r$.[34]

B.5 Derivation of Approximations

We derive the approximation $\psi^a(t, o)$ by first taking a first-order Taylor approximation of $\psi(t, o)$ around $t = 0$.

Recall that:

$$\phi(0) = \frac{1}{\sqrt{2\pi}}, \qquad \Phi(0) = \frac{1}{2}.$$

For $o = +$,

$$\psi(0, +) = \frac{\phi(0)}{\Phi(0)} = \frac{(1/\sqrt{2\pi})}{1/2} = \sqrt{\frac{2}{\pi}}.$$

For $o = -$, use $\phi(-t) = \phi(t)$ and $\Phi(-t) = 1 - \Phi(t)$:

$$\psi(0, -) = -\frac{\phi(-0)}{\Phi(-0)} = -\frac{\phi(0)}{\Phi(0)} = -\sqrt{\frac{2}{\pi}}.$$

Thus,

$$\psi(0, o) = \operatorname{sgn}(o)\sqrt{\frac{2}{\pi}} \tag{16}$$

Next we take the derivative of $\psi(t, +)$ with respect to t. Using the quotient rule and the fact

[34] For example, if $r = \mu_r = 0$ and $\mu_x = 2$, $x = 0.1$, $\sigma_o = 1$, $\sigma_r^2 = 0$, $\sigma_p^2 = 0.04$, $\sigma_e^2 = 4$. Then $C < 0$ (numerically ≈ -0.0402).

that $\phi'(t) = -t\phi(t)$ and $\Phi'(t) = \phi(t)$,

$$\begin{aligned}
\psi_t(t,+) &= \frac{\phi'(t)\Phi(t) - \phi(t)\Phi'(t)}{\Phi(t)^2} \\
&= \frac{(-t\phi(t))\Phi(t) - \phi(t)\phi(t)}{\Phi(t)^2} \\
&= -\frac{\phi(t)}{\Phi(t)}\left(t + \frac{\phi(t)}{\Phi(t)}\right) \\
&= -\psi(t,+)\big(t + \psi(t,+)\big).
\end{aligned}$$

Evaluating this at $t = 0$ gives

$$\psi_t(0,+) = -\frac{2}{\pi}. \tag{17}$$

A similar argument shows that, for $o = -$ it is also the case that

$$\psi_t(0,-) = -\frac{2}{\pi}. \tag{18}$$

Thus we can approximate

$$\begin{aligned}
\psi(t,o) &\approx \psi(0,o) + \psi_t(0,o)\,t \equiv \hat{\psi}^a(t,o) \\
&= \operatorname{sgn}(o)\sqrt{\frac{2}{\pi}} - \frac{2}{\pi}.t
\end{aligned}$$

Note, however that this approximation will result in $\hat{\psi}^a(t,+) < 0$ and $\hat{\psi}^a(t,+) > 0$ for t of large enough magnitude. Given that we know that $\psi(t,+) \geq 0$ and $\psi(t,-) \leq 0$, we can add the restriction that we wish the approximation to also have these signs. We can define our desired approximation pointwise as the function that is closest to $\hat{\psi}^a(t,.)$ while respecting these sign restrictions, i.e.

$$\begin{aligned}
\psi^a_-(t,+) &= \arg\min_{a\in\mathbb{R}^+}\left(a - \hat{\psi}^a(t,+)\right)^2 \\
\psi^a_-(t,-) &= \arg\min_{a\in\mathbb{R}^-}\left(a - \hat{\psi}^a(t,-)\right)^2
\end{aligned}$$

This leads to

$$\begin{aligned}
\psi^a_-(t) &= \max\left\{\sqrt{\frac{2}{\pi}} - \frac{2}{\pi}.t, 0\right\} \\
\psi^a_-(t) &= -\max\left\{\sqrt{\frac{2}{\pi}} + \frac{2}{\pi}.t, 0\right\}
\end{aligned}$$

Note that $\psi^a(t,o)$ is a better approximation for $\psi(t,o)$ than $\hat{\psi}^a(t,o)$, in the sense that, for every t and o

$$|\psi^a(t,o) - \psi(t,o)| \leq |\hat{\psi}^a(t,o) - \psi(t,o)|$$

with the inequality strict for some values.

Next we derive an expression for $\overline{\psi}^a(\cdot)$, used in the approximation of the average boost across signal realizations, $\mathbb{E}[\tilde{u}_x^{s,o}]$.

As we show in the proof of Proposition 13

$$\begin{aligned} & E_{s,o}\psi(z_s,o)] \\ = \quad & E_s\left[\Phi\left(\frac{u_x-u_r}{\sigma_o}\right)\psi(z_s,+)+\left(1-\Phi\left(\frac{u_x-u_r}{\sigma_o}\right)\right)\psi(z_s,-)\right] \end{aligned}$$

We first approximate $\psi(z_s,+)$ and $\psi(z_s,-)$ with $\psi^a_+(z_s)$ and $\psi^a_-(z_s)$. We then can proceed in one of two ways. First, we can approximate ψ^a_+ and ψ^a_- as linear functions, and so replace $E_s\psi(z_s,+)$ and $E_s\psi(z_s,-)$ with $\psi^a_+(t)$ and $\psi^a_-(t)$ respectively, where $t = \frac{\lambda u_x+(1-\lambda)\tilde{u}_x-\bar{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_0^2}}$. This gives the approximation

$$E_{s,o}(\psi(z_s,o)) \simeq \left[\Phi\left(\frac{u_x-u_r}{\sigma_o}\right)\psi^a_+(t)+\left(1-\Phi\left(\frac{u_x-u_r}{\sigma_o}\right)\right)\psi^a_-(t)\right]$$

An alternative is to recognize that

$$\psi^a_+(z_s) = \max\left\{\sqrt{\frac{2}{\pi}}-\frac{2}{\pi}z_s, 0\right\}$$

is the censored version of an underlying normal variable with a mean $\bar{\psi}^s_+ = \sqrt{\frac{2}{\pi}} - \frac{2}{\pi}\frac{u_x-\bar{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_0^2}}$ and variance $\sigma^2_\psi = \frac{4}{\pi^2}\frac{\lambda^2}{\sigma_x^2+\sigma_r^2+\sigma_0^2}$. The formula for the mean of a normal censored below at zero gives

$$E_S\left(\psi^a_+(z_s)\right) = \bar{\psi}^s_+\Phi\left(\frac{\bar{\psi}^s_+}{\sigma_\psi}\right)+\sigma_\psi\phi\left(\frac{\bar{\psi}^s_+}{\sigma_\psi}\right)$$

In contrast

$$\psi^a_-(z_s) = -\max\left\{\sqrt{\frac{2}{\pi}}+\frac{2}{\pi}z_s, 0\right\}$$

and so, with $\bar{\psi}_{-}^{s} = \sqrt{\frac{2}{\pi}} + \frac{2}{\pi}\frac{u_x - \bar{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_0^2}}$ we have

$$E_S\left(\psi_{-}^{a}(z_s)\right) = -\left(\bar{\psi}_{-}^{s}\Phi\left(\frac{\bar{\psi}_{-}^{s}}{\sigma_\psi}\right) + \sigma_\psi \phi\left(\frac{\bar{\psi}_{-}^{s}}{\sigma_\psi}\right)\right)$$

giving

$$\begin{aligned} E_{s,o}(\psi(z_s, o)) \quad = & \\ & \Phi\left(\frac{u_x - u_r}{\sigma_o}\right)\left(\bar{\psi}_{+}^{s}\Phi\left(\frac{\bar{\psi}_{+}^{s}}{\sigma_\psi}\right) + \sigma_\psi \phi\left(\frac{\bar{\psi}_{+}^{s}}{\sigma_\psi}\right)\right) \\ & -\left(1 - \Phi\left(\frac{u_x - u_r}{\sigma_o}\right)\right)\left(\bar{\psi}_{-}^{s}\Phi\left(\frac{\bar{\psi}_{-}^{s}}{\sigma_\psi}\right) + \sigma_\psi \phi\left(\frac{\bar{\psi}_{-}^{s}}{\sigma_\psi}\right)\right) \end{aligned}$$

B.6 Further Details on Compromise Effect

Our model can generate either the compromise effect or the opposite, depending on where the decoy is placed.

As a concrete example, consider a case in which interim beliefs are such that the means of x are {2,2} on dimensions 1 and 2 respectively, while the means of y are {3,1}. Assume that standard deviations are 1 for each alternative on each dimension. If we add a phantom decoy z with means {1,3} (and standard deviations also equal to 1) then this will have no effect on the probability of choice between x and y due to the symmetry of the problem.

If instead the z has mean {1,4} then our model predicts the decoy effect. This is because moving z further away in dimension 2 has a bigger effect on the closer option. In dimension 1, x receives 0 boost, as the positive boost from z and the negative boost from y cancel each other out. Using the formula from Balakrishnan and Dean (2025), y receives a positive boost of about 0.34. In dimension 2, x receives a positive boost of 0.18: the positive boost from y is smaller than the negative boost from z because the former is closer. y receives a negative boost of 0.31, meaning that x is favored overall.

If instead we place the decoy at {0,3} the pattern is reversed. Now x receives a negative boost in dimension 1 of -0.18 and a 0 boost in dimension 2, while y receives a positive boost of 0.31 in dimension 1 and a negative boost of 0.34 in dimension 2, meaning that overall the addition of z favors y.

B.7 Range Effects with Ordinal Noise

We now generalize our results on range effects to the case of ordinal noise. We have seen how, without ordinal noise, our model generates a range-contrast effect. We now show what happens when we add ordinal noise. Let us denote a *range-normalization effect* as the opposite of a range-contrast effect.[35]

With ordinal noise, another force is at play. To illustrate it, suppose we have substantial noise in the cardinal dimension—to build intuition, suppose we get no cardinal information ($\sigma_\epsilon^2 \to \infty$). If we increase $u_{x,1}$, the only change is that it decreases how often we get the incorrect ordinal signals $u_{x,1} > u_{z,1}$ and $u_{y,1} > u_{z,1}$. However, the first (which inflates the beliefs about $u_{x,1} - u_{y,1}$) is always more sensitive to $u_{z,1}$ than the second (because $u_{x,1}$ is closer), so the net effect is always downward. When cardinal information is present, this force is damped, as the ceiling effects we discussed earlier are now also present, and we get a non-monotonicity: range-normalization when $u_{z,1}$ is close to $u_{x,1}$, and range-contrast when $u_{z,1}$ is large enough.

The following proposition, which generalizes Proposition 8 in the main body, formalizes this discussion and provides complete results.

Proposition 16 (Range Effects with and without Ordinal Noise.)**.** *Consider x, y and z such that $u_{x,m} > u_{y,m}$ for some $m \in \{1, \dots, n\}$, and $u_{z,m}$ is not between $u_{x,m}$ and $u_{y,m}$. Suppose we have the same prior and signal precisions in all dimensions ($\tilde{u}_{j,i}$, $\sigma^2_{p_{j,i}} > 0$, $\sigma^2_{\epsilon_{j,i}} > 0$ are the same for $j = x, y, z$ and $i = 1, \dots n$, and σ_o^2 is the same for all binary comparisons), and let s and o denote all cardinal and ordinal signals. Then*

(a) If $\sigma_o^2 = 0$, then beliefs exhibit a range-contrast effect on dimension m at $\{x, y, z\}$.

(b) If $\sigma_o^2 > 0$ and $\sigma_\epsilon^2 \to \infty$, then beliefs exhibit a range-normalization effect on dimension m at $\{x, y, z\}$.

(c) If $\sigma_o^2 > 0$, then there exists $\bar{\sigma}^2$ such that, if $\sigma_\epsilon^2 > \bar{\sigma}^2$, beliefs exhibit a range-normalization effect on dimension m at $\{x, y, z\}$ when $u_{z,m}$ is close to $[u_{y,m}, u_{x,m}]$ and a range-contrast effect as it gets further.

[35] Formally, we say that beliefs exhibit a *range-normalization effect on dimension m at $\{x, y, z\}$* if the following holds. Let s and o denote all ordinal and cardinal signals. Then, for any $i, j \in x, y, z$ with if $u_{i,m} > u_{j,m}$, letting $l := \{x, y\} \backslash \{i, j\}$,

(i) If $u_{l,m} > u_{i,m} > u_{j,m}$, then $\mathbb{E}\left[\mathbb{E}\left[U(i) - U(j) | s, o\right]\right]$ is strictly decreasing in $u_{k,1}$;

(ii) If $u_{i,1} > u_{j,1} > u_{l,1}$, then $\mathbb{E}\left[\mathbb{E}\left[U(i) - U(j) | s, o\right]\right]$ is strictly increasing in $u_{k,1}$.

C Caution

In this section we expand on the cautious extension of our model discussed in Section 7.

As mentioned above, we model aversion to uncertainty using the classical approach: adding a concave transformation, which yields aversion to uncertainty exactly like the concavity of a utility function leads to risk aversion in standard Expected Utility; or like a concave transformation leads to ambiguity aversion in the Smooth model of ambiguity aversion (Klibanoff et al., 2005). In particular, let $\rho : \mathbb{R} \to \mathbb{R}$, strictly increasing and continuous. Then, if x is the item of interest with $U(x) = \sum u_{x,i}$ its utility distributed according to $p_{U(x)}$ (under the assumptions of the paper) and $\mathcal{I}$ a (Borel) information set, denote by M the set of all combinations of $x|\mathcal{I}$ that we study.[36] Then, the agent evaluates each $x|\mathcal{I} \in M$ by

$$V(x|\mathcal{I}) = \int \rho\left(U(x)\right) \mathrm{d}p_{U(x)|\mathcal{I}}. \tag{19}$$

When preferences are represented as in (19) for all combinations, we say that it admits a *cautious representation*.[37] Note that this model is very general and requires very few assumptions: it is a routine exercise to show that preferences can be represented this way if and only if they are complete preferences that satisfy monotonicity (higher utility is better), (topological) continuity, and standard vNM Independence.

For the purposes of our paper, particularly natural is the case in which ρ is of the CARA form, that is $\rho(x) := -e^{-\alpha x}/\alpha$ for $\alpha \neq 0$, and $\rho(x) = x$ for $\alpha = 0$; α captures the attitude towards this uncertainty—$\alpha > 0$ captures aversion, $\alpha = 0$ neutrality, and $\alpha < 0$ an uncertainty seeking attitude.[38] Importantly, this gives a very tractable functional form in our setup, as the following proposition illustrates.

Proposition 17. *Let $\succcurlyeq$ be a preference relation over M that admits a cautious representation with*

[36]Recall that we assume that $U(x) = \sum u_{x,i}$ where each $u_{x,i}$ are independent, and that we only consider information sets $\mathcal{I}$ that can be written independently across dimensions.

[37]A preference relation is a symmetric and transitive binary relation. As noted above, this representation is related to the one in Cerreia-Vioglio et al. (2024). Indeed, notice how the uncertainty is about the utilities in each dimension, and the min form of the representation in that paper can be seen as a limit case of aversion, just like the MaxMin Expected Utility model of (Gilboa and Schmeidler, 1989) can be seen as a limit case of the Smooth ambiguity preferences of (Klibanoff et al., 2005). However, as noted earlier, unlike that paper, here we do not consider options relative to a reference point, but in absolute terms, and study the impact of information.

[38]The CARA version of the model is characterized by the following behavioral property. Denote for any $x \in M$ and any $m \in \mathbb{R}$, denote $m_1 x$ the object that returns (known) utility m in dimension 1 and x otherwise. Then, we say that a preference is *separable* if $m_1 x \succcurlyeq m_1 y \Leftrightarrow n_1 x \succcurlyeq n_1 y$ for all $x, y \in M$ and $m, n \in \mathbb{R}$. It is a routine exercise to show that if $\succcurlyeq$ admits a cautious representation with index ρ, then $\succcurlyeq$ is separable if and only if $\exists \alpha \in \mathbb{R}$ such that $\rho(x) = -e^{-\alpha x}$.

$\rho = -e^{-\alpha x}/\alpha$ with $\alpha \neq 0$. Then $\succcurlyeq$ is represented by

$$\bar{V}(x|\mathcal{I}) := \sum_{i}^{n} \mathrm{CE}(u_{x,i}|\mathcal{I}) \qquad \textit{where} \qquad \mathrm{CE}(u_{x,i}|\mathcal{I}) := -\frac{1}{\alpha} \ln \mathbb{E}\left[e^{-\alpha u_{x,i}}|\mathcal{I}\right]. \tag{20}$$

If $u_x \sim N(\tilde{u}_x, \sigma_x^2)$, then

$$\mathrm{CE}(u_x|s) = \tilde{u}_x^s - \alpha \frac{\sigma_x^2}{2}.$$

If, in addition, we also have $u_r \sim N(\tilde{u}_r, \sigma_r^2)$ and $o = +$ iff $x > r + v$ with $v \sim N(0, \sigma_o^2)$, with all random variables independent, then

$$\mathrm{CE}(x|s,o) = \tilde{u}_x^s - \alpha\frac{\sigma_x^2}{2} + \bar{\psi}\left(\frac{\tilde{u}_x^s - \tilde{u}_r}{\sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}}, \frac{\alpha\sigma_x^2}{\sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}}, o\right) \qquad \bar{\psi}(m, \alpha k, o) := \begin{cases} \frac{1}{\alpha}\left(\ln\Phi(m) - \ln\Phi(m - \alpha k)\right) \\ -\frac{1}{\alpha}\left(\ln\Phi(-m + \alpha k) - \ln\Phi(-m)\right) \end{cases}$$

This proposition shows that preferences admit a very tractable representation, even if we add caution. First, it shows that we can represent aggregate preferences as the sum of the dimension-by-dimension *certainty equivalents* $\mathrm{CE}(u_{x,i}|\mathcal{I})$.[39] Moreover, each dimension's component admits a simple functional form. To understand it, let us compare with the mean beliefs after ordinal signals obtained in Proposition 1. Suppose that $o = +$ and recall from Proposition 1 that

$$\mathbb{E}[u_x \mid +, s] = \underbrace{\tilde{u}_x^s}_{\text{Ref.-indep. component}} + \underbrace{\frac{\sigma_x^2}{\sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}} \cdot \frac{\phi\left(\frac{\tilde{u}_x^s - \tilde{u}_r}{\sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}}\right)}{\Phi\left(\frac{\tilde{u}_x^s - \tilde{u}_r}{\sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}}\right)}}_{\text{Boost} > 0}.$$

With caution, we replace this with

$$\mathrm{CE}(x \mid s, o) = \underbrace{\tilde{u}_x^s}_{\text{Ref.-indep.}} - \underbrace{\alpha\frac{\sigma_x^2}{2}}_{\text{Variance adjustment}} + \underbrace{\frac{1}{\alpha}\left(\ln\Phi\left(\frac{\tilde{u}_x^s - \tilde{u}_r}{\sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}}\right) - \ln\Phi\left(\frac{\tilde{u}_x^s - \tilde{u}_r - \alpha\sigma_x^2}{\sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}}\right)\right)}_{\alpha\text{-Adjusted boost} > 0}.$$

That is, adding caution implies *i*) adding a variance punishment, using the variance *before* the ordinal signal (which is simple to compute); and *ii*) change the boost to a α-dependent formula. Note that the α-adjusted boost maintains the expected signs, and, in particular, it is strictly

[39]Indeed, note that $\mathrm{CE}(u_{x,i}|\mathcal{I})$ is the certainty equivalent of $u_{x,i}|\mathcal{I}$ according to the agent's preferences. Note that using certainty equivalents for doing dimension-by-dimension aggregation not only is the result given by the proposition, but it is also the natural step—if we were want to aggregate across dimensions, we need to make sure items are normalized to the same unit of measure, and certainty equivalents guarantee this is the case.

positive $o = +$ and strictly negative otherwise.[40] Indeed, it is also easily verified that this modified boost coincides with that identified in Proposition 1 if $\alpha = 0$, that is, $\lim_{\alpha \downarrow 0} \bar{\psi}(m, \alpha k, +) = k\,\frac{\phi(m)}{\Phi(m)}$.[41] Moreover, the boost maintains the core properties that we identified for mean beliefs. In addition, it is also easy check that, unless beliefs are degenerate, the CE is *decreasing* in α; while the boost itself increases with α (the ordinal correction is amplified), this is dominated by the growing variance penalty.

Two more proporties are worth highlighting. First, despite aversion to uncertainty, preferences that admit a cautious representation continue to satisfy FOSD, that is, strictly prefer a choice that first order stochastically dominates another: this is trivially true, since they are akin to Expected Utility with a strictly increasing index ϕ, which are known to satisfy FOSD. Second, it is easily verified that, unless beliefs are degenerate, values in CE—and therefore of $\bar{V}$—are strictly decreasing in α whenever beliefs are not degenerate.[42] That is: risk aversion lowers the values in each dimension, as intuitive. We are now ready to study the implications of caution in our context.

The Endowment Effect and Status Quo Bias. Consider the classic evidence of the endowment effect; here, we review and expand on the setup discussed in the main body. For a given object x of true utility $u_x \in \mathbb{R}_{++}$, the DM needs to compute: the price $p_{\mathrm{WTP}} > 0$ they are willing to pay to acquire the object when they do not own it; the price $p_{\mathrm{WTA}} > 0$ they are willing to accept to sell the object if they own it; the price $p_{\mathrm{Ch}} > 0$ that make them indifferent between that price and the object in a choice. We follow the assumptions we held throughout the paper: the agent has a prior $N(\mu_x, \sigma^2_{p,x})$ for u_x, of $N(\mu_n, \sigma^2_{p,n})$ for the utility u_n of not having it, and of $N(\mu_y, \sigma^2_{p,m})$ on monetary exchanges of \$$y$; receives cardinal signals with variance σ^2_ϵ on all unknowns; receives all ordinal signals in each dimension, with ordinal noise σ^2_o. Moreover, as discussed above, we assume that they receive an additional cardinal signal about the value of

[40] Suppose $\alpha > 0$ and $o = +$. Then, the boost is obviously positive since $\alpha\sigma^2_x > 0$ and both Φ and ln are strictly increasing. When $\alpha < 0$, then $\alpha\sigma^2_x < 0$ but also $\frac{1}{\alpha} < 0$. The case $o = -$ is symmetric.

[41] To see why, suppose $o = +$ and let $g(z) := \ln \Phi(z)$, so $g'(z) = \phi(z)/\Phi(z)$ and g is C^1 on $\mathbb{R}$. By the mean value theorem, for each $\alpha > 0$ there exists $\xi_\alpha \in (m - \alpha k, m)$ such that

$$\bar{\psi}(m, \alpha k, +) = \frac{g(m) - g(m - \alpha k)}{\alpha} = k\, g'(\xi_\alpha) = k\, \frac{\phi(\xi_\alpha)}{\Phi(\xi_\alpha)}.$$

Since $\xi_\alpha \to m$ as $\alpha \downarrow 0$ and ϕ/Φ is continuous, $\lim_{\alpha \downarrow 0} \bar{\psi}(m, \alpha k, +) = k\, \frac{\phi(m)}{\Phi(m)}$.

[42] To see why, recall that we can rewrite CE using Gibbs variational representation, that is, it is know that

$$\mathrm{CE}_\alpha(X \mid \mathcal{I}) = \inf_{Q \ll P(\cdot \mid \mathcal{I})} \left\{ \mathbb{E}_Q[X] + \frac{1}{\alpha} D(Q \| P(\cdot \mid \mathcal{I})) \right\},$$

where $D(\cdot \| \cdot)$ denotes relative entropy. Since the entropy term is nonnegative, increasing α lowers the weight $1/\alpha$ and therefore weakly reduces the infimum.

the endowment with noise σ_e^2.To highlight which effects are due to our boosts and caution, we also assume that the true utility of money is the identity function, that all priors are well calibrated ($\mu_x = u_x$, $\mu_r = u_r$, $\mu_y = y$), that the utility of not having the object is 0, that the priors of owning or not owning an object have the same variance ($\sigma_{p,x}^2 = \sigma_{p,n}^2$), and that there is no ordinal noise ($\sigma_o^2 = 0$).[43] Moreover, in what follows, we focus on the case in which all cardinal signals are received at their expected value (no noise, the modal outcome), noting that our conclusions about the endowment effect hold in general. As in the rest of the paper, let σ_x^2 denote the posterior variance about u_x (and u_n) when it is not the reference point, and $\hat{\sigma}_x^2$ when it is; σ_m^2 and $\hat{\sigma}_m^2$ denote the corresponding quantities for the utility of \$y.

We can now simply apply our formulas. When computing the Willingness to Pay p_{WTP}, the agent is equating the value of not having the object but also paying no money (the endowment), and that of acquiring the object and paying $p_{\text{WTP}} > 0$. This means

$$\underbrace{0 - \frac{\alpha}{2}\hat{\sigma}_x^2 + \bar{\psi}\left(\frac{-u_x}{\sqrt{\hat{\sigma}_x^2 + \sigma_x^2}}, \frac{\alpha\hat{\sigma}_x^2}{\sqrt{\hat{\sigma}_x^2 + \sigma_x^2}}, -\right)}_{\text{Value of No Object, Boosted Down}} + \underbrace{0 - \frac{\alpha}{2}\hat{\sigma}_m^2 + \bar{\psi}\left(\frac{p_{\text{WTP}}}{\sqrt{\hat{\sigma}_m^2 + \sigma_m^2}}, \frac{\alpha\hat{\sigma}_m^2}{\sqrt{\hat{\sigma}_m^2 + \sigma_m^2}}, +\right)}_{\text{Value of \$0, Boosted Up}} =$$

$$\underbrace{u_x - \frac{\alpha}{2}\sigma_x^2 + \bar{\psi}\left(\frac{u_x}{\sqrt{\hat{\sigma}_x^2 + \sigma_x^2}}, \frac{\alpha\sigma_x^2}{\sqrt{\hat{\sigma}_x^2 + \sigma_x^2}}, +\right)}_{\text{Value of Buying Object, Boosted Up}} - \underbrace{p_{\text{WTP}} - \frac{\alpha}{2}\sigma_m^2 + \bar{\psi}\left(\frac{-p_{\text{WTP}}}{\sqrt{\hat{\sigma}_m^2 + \sigma_m^2}}, \frac{\alpha\sigma_m^2}{\sqrt{\hat{\sigma}_m^2 + \sigma_m^2}}, -\right)}_{\text{Value of paying } p_{\text{WTP}}\text{, Boosted Down}}.$$

which gives[44]

$$\begin{aligned}
p_{\text{WTP}} &= u_x - \frac{\alpha}{2}\left(\left(\sigma_x^2 - \hat{\sigma}_x^2\right) + \left(\sigma_m^2 - \hat{\sigma}_m^2\right)\right) + \psi\left(\frac{u_x}{\sqrt{\hat{\sigma}_x^2 + \sigma_x^2}}, \frac{\alpha\sigma_x^2}{\sqrt{\hat{\sigma}_x^2 + \sigma_x^2}}, +\right) - \psi\left(\frac{-u_x}{\sqrt{\hat{\sigma}_x^2 + \sigma_x^2}}, \frac{\alpha\hat{\sigma}_x^2}{\sqrt{\hat{\sigma}_x^2 + \sigma_x^2}}, -\right) \\
&\quad + \bar{\psi}\left(\frac{-p_{\text{WTP}}}{\sqrt{\hat{\sigma}_m^2 + \sigma_m^2}}, \frac{\alpha\sigma_m^2}{\sqrt{\hat{\sigma}_m^2 + \sigma_m^2}}, -\right) - \bar{\psi}\left(\frac{p_{\text{WTP}}}{\sqrt{\hat{\sigma}_m^2 + \sigma_m^2}}, \frac{\alpha\hat{\sigma}_m^2}{\sqrt{\hat{\sigma}_m^2 + \sigma_m^2}}, +\right) \\
&= u_x - \frac{\alpha}{2}\left(\left(\sigma_x^2 - \hat{\sigma}_x^2\right) + \left(\sigma_m^2 - \hat{\sigma}_m^2\right)\right) + \frac{1}{\alpha}\ln\frac{\Phi\left(\frac{u_x + \alpha\hat{\sigma}_x^2}{\sqrt{\hat{\sigma}_x^2 + \sigma_x^2}}\right)\Phi\left(\frac{p_{\text{WTP}} - \alpha\hat{\sigma}_m^2}{\sqrt{\hat{\sigma}_m^2 + \sigma_m^2}}\right)}{\Phi\left(\frac{u_x - \alpha\sigma_x^2}{\sqrt{\hat{\sigma}_x^2 + \sigma_x^2}}\right)\Phi\left(\frac{p_{\text{WTP}} + \alpha\sigma_m^2}{\sqrt{\hat{\sigma}_m^2 + \sigma_m^2}}\right)}.
\end{aligned} \tag{21}$$

To compute the Willingness to Accept p_{WTA}, the agent is equating the value of keeping the

[43]Note how Bayesian shrinkage may trivially generate the endowment effect or its opposite, e.g., when the prior of the object is below the true value; as this pertains to foreces unrelated to our analysis, we rule this out.

[44]While this is an implicit equation, the observation below show that it admits a unique solution. The same holds for the equations for p_{WTA} and p_{Ch} below.

object but also getting no money (the endowment), and that of foregoing the object and receiving $p_{\text{WTA}} > 0$. This means

$$\underbrace{u_x - \frac{\alpha}{2}\hat{\sigma}_x^2 + \bar{\psi}\left(\frac{u_x}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, \frac{\alpha\hat{\sigma}_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, +\right)}_{\text{Value of Keeping Object, Boosted Up}} \underbrace{+0 - \frac{\alpha}{2}\hat{\sigma}_m^2 + \bar{\psi}\left(\frac{-p_{\text{WTA}}}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, \frac{\alpha\hat{\sigma}_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, -\right)}_{\text{Value of \$0, Boosted Down}} =$$

$$\underbrace{0 - \frac{\alpha}{2}\sigma_x^2 + \bar{\psi}\left(\frac{-u_x}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, \frac{\alpha\sigma_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, -\right)}_{\text{Value of No Object, Boosted Down}} \underbrace{+p_{\text{WTA}} - \frac{\alpha}{2}\sigma_m^2 + \bar{\psi}\left(\frac{p_{\text{WTA}}}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, \frac{\alpha\sigma_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, +\right)}_{\text{Value of receiving } p_{\text{WTA}}\text{, Boosted Up}}.$$

This gives

$$\begin{aligned}
p_{\text{WTA}} &= u_x + \frac{\alpha}{2}\left(\left(\sigma_x^2 - \hat{\sigma}_x^2\right) + \left(\sigma_m^2 - \hat{\sigma}_m^2\right)\right) - \bar{\psi}\left(\frac{-u_x}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, \frac{\alpha\sigma_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, -\right) + \bar{\psi}\left(\frac{u_x}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, \frac{\alpha\hat{\sigma}_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, +\right) \\
&\quad + \bar{\psi}\left(\frac{-p_{\text{WTA}}}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, \frac{\alpha\hat{\sigma}_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, -\right) - \bar{\psi}\left(\frac{p_{\text{WTA}}}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, \frac{\alpha\sigma_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, +\right) \\
&= u_x + \frac{\alpha}{2}\left(\left(\sigma_x^2 - \hat{\sigma}_x^2\right) + \left(\sigma_m^2 - \hat{\sigma}_m^2\right)\right) + \frac{1}{\alpha}\ln\frac{\Phi\left(\frac{u_x+\alpha\sigma_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}\right)\Phi\left(\frac{p_{\text{WTA}}-\alpha\sigma_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}\right)}{\Phi\left(\frac{u_x-\alpha\hat{\sigma}_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}\right)\Phi\left(\frac{p_{\text{WTA}}+\alpha\hat{\sigma}_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}\right)}.
\end{aligned} \tag{22}$$

To compute p_{Ch}, the agent is equating the value of obtaining the object vs. obtaining p_{Ch}, where neither is the endowment. This means

$$\underbrace{u_x - \frac{\alpha}{2}\sigma_x^2 + \bar{\psi}\left(\frac{u_x}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, \frac{\alpha\sigma_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, +\right)}_{\text{Value of Receiving the Object, Boosted Up}} \underbrace{+0 - \frac{\alpha}{2}\hat{\sigma}_m^2 + \bar{\psi}\left(\frac{-p_{\text{Ch}}}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, \frac{\alpha\hat{\sigma}_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, -\right)}_{\text{Value of \$0, Boosted Down}} =$$

$$\underbrace{0 - \frac{\alpha}{2}\hat{\sigma}_x^2 + \bar{\psi}\left(\frac{-u_x}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, \frac{\alpha\hat{\sigma}_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, -\right)}_{\text{Value of No Object, Boosted Down}} \underbrace{+p_{\text{Ch}} - \frac{\alpha}{2}\sigma_m^2 + \bar{\psi}\left(\frac{p_{\text{WTA}}}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, \frac{\alpha\sigma_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, +\right)}_{\text{Value of receiving } p_{\text{Ch}}\text{, Boosted Up}}.$$

This gives

$$
\begin{aligned}
p_{\mathrm{Ch}} = u_x + \frac{\alpha}{2}\left(\left(\sigma_m^2 - \hat{\sigma}_m^2\right) - \left(\sigma_x^2 - \hat{\sigma}_x^2\right)\right) + \bar{\psi}\left(\frac{u_x}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, \frac{\alpha\sigma_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, +\right) - \bar{\psi}\left(\frac{-u_x}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, \frac{\alpha\hat{\sigma}_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}, -\right) \\
+ \bar{\psi}\left(\frac{-p_{\mathrm{Ch}}}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, \frac{\alpha\hat{\sigma}_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, -\right) - \bar{\psi}\left(\frac{p_{\mathrm{Ch}}}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, \frac{\alpha\sigma_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}, +\right) \\
= u_x + \frac{\alpha}{2}\left(\left(\sigma_m^2 - \hat{\sigma}_m^2\right) - \left(\sigma_x^2 - \hat{\sigma}_x^2\right)\right) + \frac{1}{\alpha}\ln\frac{\Phi\left(\frac{u_x+\alpha\hat{\sigma}_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}\right)\Phi\left(\frac{p_{\mathrm{Ch}}-\alpha\sigma_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}\right)}{\Phi\left(\frac{u_x-\alpha\sigma_x^2}{\sqrt{\sigma_x^2+\hat{\sigma}_x^2}}\right)\Phi\left(\frac{p_{\mathrm{Ch}}+\alpha\hat{\sigma}_m^2}{\sqrt{\hat{\sigma}_m^2+\sigma_m^2}}\right)}. \qquad (23)
\end{aligned}
$$

The following observation, proved below, shows that indeed we have the endowment effect—and p_{Ch} in between, as long as there is uncertainty about the value of the object. The difference between p_{Ch} and p_{WTP}, instead, relies on the uncertainty about the value of money.

Proposition 18. *Consider the setup above and $p_{\mathrm{WTA}}, p_{\mathrm{WTP}}, p_{\mathrm{Ch}}$ defined in Eq.* (22)-(23). *Then:*

1. *$p_{\mathrm{WTA}}, p_{\mathrm{WTP}}, p_{\mathrm{Ch}}$ are well-defined, that is, Equations* (22)-(23) *admit a unique solution.*
2. *Suppose $\alpha, \sigma_x^2, \sigma_e^2 > 0$. Then:*
   (a) *$p_{\mathrm{WTA}} > p_{\mathrm{WTP}}$;*
   (b) *Moreover:*
      i. *If there is uncertainty about the value of money ($\sigma_m^2 > 0$), $p_{\mathrm{WTA}} > p_{\mathrm{Ch}} > p_{\mathrm{WTP}}$;*
      ii. *If there is no uncertainty about the value of money ($\sigma_m^2 = 0$), $p_{\mathrm{WTA}} > p_{\mathrm{Ch}} = p_{\mathrm{WTP}}$.*

Figure 7 shows how the endowment effect value uncertainty σ_x^2 and with u_x—showing how it is indeed increasing in the degree of uncertainty about the value of the object.

Lastly, as already mentioned in the main body, note how an identical line of argument can be adapted to show how the model easily generates status quo bias.

D Proofs of the Results in the Text

D.1 Proof of Proposition 1.

This now follows as an immediate corollary of Proposition 10, setting the covariance between x and r to zero.

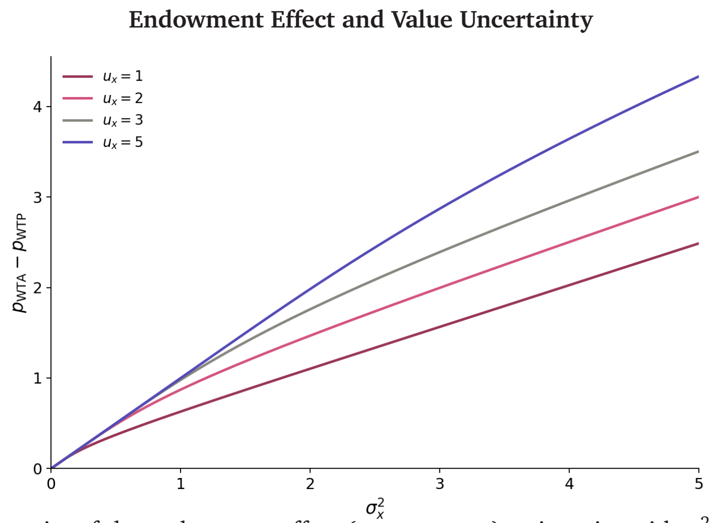

Figure 7: Comparative statics of the endowment effect ($p_{\mathrm{WTA}} - p_{\mathrm{WTP}}$) as it varies with σ_x^2, following the formulas in this section. These are drawn assuming $\sigma_m^2 \hat{\sigma}_x^2 = 0$ and alpha $=1$.

D.2 Proof of Proposition 2

Assume $\tilde{u}_x = \tilde{u}_r = u_r$ and $\sigma_p^2, \sigma_\varepsilon^2 > 0$ and let

$$\delta := u_x - u_r, \qquad \lambda := \frac{\sigma_p^2}{\sigma_p^2 + \sigma_\varepsilon^2} \in (0,1), \qquad \sigma_x^2 := \frac{\sigma_p^2 \sigma_\varepsilon^2}{\sigma_p^2 + \sigma_\varepsilon^2}, \qquad \tau := \sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}.$$

By Proposition 1 ,

$$M(\delta) := \mathbb{E}[\tilde{u}_x^{s,o}] = u_r + \lambda\delta + \frac{\sigma_x^2}{\tau}\,\mathcal{B}(\delta), \tag{24}$$

where the *average boost* is

$$\mathcal{B}(\delta) = p(\delta)\,\mathbb{E}_\varepsilon[\rho(\eta)] - \big(1 - p(\delta)\big)\,\mathbb{E}_\varepsilon[\rho(-\eta)], \tag{25}$$

with $p(\delta) := \Phi(\delta/\sigma_o)$, $\eta := \lambda(\delta + \varepsilon)/\tau \sim N(m, v^2)$, $m := \lambda\delta/\tau$, $v := \lambda\sigma_\varepsilon/\tau$, and $\rho(t) := \phi(t)/\Phi(t)$ the inverse Mills ratio (IMR). We rely on the following well-known properties of the IMR (the proof is standard and can be found in several textbooks).

Claim 1. *For all $t \in \mathbb{R}$:*

(i) $\rho(t) > 0$.

(ii) $\rho'(t) = -\rho(t)\big(\rho(t) + t\big)$ *and, moreover,* $-1 < \rho'(t) < 0$ *for all* $t \in \mathbb{R}$.

(iii) $\rho(t) \to 0$ *as* $t \to +\infty$, *and* $\rho(t) + t \to 0$ *as* $t \to -\infty$.

(iv) $\rho(0) = \sqrt{2/\pi}$ *and* $\rho'(0) = -(2/\pi)$.

(v) ρ *is convex on* $\mathbb{R}$.

(vi) *For* $t > 0$, $\rho(t) \le \frac{1}{t}$ *and* $\rho(-t) \le t + \frac{1}{t}$.

Throughout, we abbreviate $\mathbb{E}_\varepsilon[\,\cdot\,]$ as $\mathbb{E}[\,\cdot\,]$, with all expectations over ε conditional on (u_x, u_r). The following additional claim will also be useful.

Claim 2. *Let*

$$H_p(z) := p\rho(z) - (1-p)\rho(-z), \qquad q_{\alpha,p}(z) := z + \alpha\big(H_p(z) + zH_p'(z)\big),$$

where $\rho(z) = \phi(z)/\Phi(z)$, $\alpha \in (0, \frac{1}{2}]$, *and* $p \in [\frac{1}{2}, 1]$. *Then:*

(i) $q_{\alpha,1/2}$ *is odd and satisfies*

$$q_{\alpha,1/2}(z) > 0 \quad \text{for all } z > 0.$$

(ii) Moreover, for each fixed z, $q_{\alpha,p}(z)$ *is strictly increasing in* p. *Consequently, if* $Z \sim N(m, v^2)$ *with* $m > 0$, *then*

$$E[q_{\alpha,p}(Z)] > 0.$$

Proof of Claim. Define

$$r(z) := \rho(z) + z\rho'(z).$$

Since

$$H_p(z) + zH_p'(z) = p\big(\rho(z) + z\rho'(z)\big) - (1-p)\big(\rho(-z) - z\rho'(-z)\big) = p\,r(z) - (1-p)\,r(-z),$$

we can rewrite

$$q_{\alpha,p}(z) = z + \alpha\big(p\,r(z) - (1-p)\,r(-z)\big). \tag{A.1}$$

We first prove (ii). Differentiating (A.1) with respect to p,

$$\frac{\partial q_{\alpha,p}(z)}{\partial p} = \alpha\big(r(z) + r(-z)\big).$$

Now

$$r(z) + r(-z) = \rho(z) + \rho(-z) + z\big(\rho'(z) - \rho'(-z)\big).$$

By Claim1(i), $\rho(z), \rho(-z) > 0$. By Claim 1(v), ρ is convex, so ρ' is increasing; hence $z(\rho'(z) - \rho'(-z)) \ge 0$ for all z. Therefore

$$r(z) + r(-z) > 0,$$

and so

$$\frac{\partial q_{\alpha,p}(z)}{\partial p} > 0 \quad \text{for all } z.$$

This proves (ii).

Next we prove (i). When $p = \frac{1}{2}$, equation (A.1) gives

$$q_{\alpha,1/2}(z) = z + \frac{\alpha}{2}\big(r(z) - r(-z)\big),$$

so $q_{\alpha,1/2}$ is odd.

It remains to show positivity for $z > 0$. Let

$$H_0(z) := H_{1/2}(z) = \frac{1}{2}\big(\rho(z) - \rho(-z)\big).$$

We claim that

$$H_0(z) > -z \qquad \text{for all } z > 0. \tag{A.2}$$

Indeed, define

$$L(z) := z + \rho(z), \qquad R(z) := \rho(-z) - z.$$

By Claim 1(ii), $-1 < \rho'(z) < 0$, so

$$L'(z) = 1 + \rho'(z) > 0, \qquad R'(z) = -1 - \rho'(-z) < 0.$$

Also $L(0) = R(0) = \rho(0)$. Hence for every $z > 0$,

$$L(z) > L(0) = R(0) > R(z),$$

that is,

$$z + \rho(z) > \rho(-z) - z,$$

which is exactly (A.2).

Moreover, again by Claim 1(ii),

$$H_0'(z) = \frac{1}{2}\big(\rho'(z) + \rho'(-z)\big) > -1. \tag{A.3}$$

Combining (A.2) and (A.3), for $z > 0$,

$$H_0(z) + zH_0'(z) > -z - z = -2z.$$

Therefore

$$q_{\alpha,1/2}(z) = z + \alpha\big(H_0(z) + zH_0'(z)\big) > z - 2\alpha z = (1-2\alpha)z \geq 0.$$

Since both inequalities above are strict, we in fact have

$$q_{\alpha,1/2}(z) > 0 \qquad \text{for all } z > 0.$$

This proves (i).

Finally, let $Z \sim N(m, v^2)$ with $m > 0$, and let f_Z denote its density. Since $m > 0$,

$$f_Z(z) > f_Z(-z) \qquad \text{for all } z > 0.$$

Using oddness and positivity of $q_{\alpha,1/2}$,

$$E[q_{\alpha,1/2}(Z)] = \int_0^\infty q_{\alpha,1/2}(z)\big(f_Z(z) - f_Z(-z)\big)\,dz > 0.$$

By part (ii), $q_{\alpha,p}(z) \geq q_{\alpha,1/2}(z)$ for all z, with strict inequality whenever $p > \frac{1}{2}$. Hence

$$E[q_{\alpha,p}(Z)] > 0.$$

■

Preliminary: derivative of the average boost. Differentiating (25) with respect to δ (using $\partial\eta/\partial\delta = \lambda/\tau$ and the Leibniz rule),

$$\mathcal{B}'(\delta) = \underbrace{\frac{\phi(\delta/\sigma_o)}{\sigma_o}\Big(\mathbb{E}[\rho(\eta)] + \mathbb{E}[\rho(-\eta)]\Big)}_{=:\,A(\delta)\,>\,0} + \underbrace{\frac{\lambda}{\tau}\Big(p(\delta)\,\mathbb{E}[\rho'(\eta)] + \big(1-p(\delta)\big)\mathbb{E}[\rho'(-\eta)]\Big)}_{=:\,D(\delta)\,<\,0}, \tag{26}$$

where $A(\delta) > 0$ by Claim 1(i) and $C(\delta) < 0$ by Claim 1(ii). From (24),

$$M'(\delta) = \lambda + \frac{\sigma_x^2}{\tau}\,\mathcal{B}'(\delta). \tag{27}$$

Part 1: Over- and under-sensitivity.

Strict monotonicity. From (27) we have $M'(\delta) = \lambda + (\sigma_x^2/\tau)\mathcal{B}'(\delta)$ with $\mathcal{B}'(\delta) = A(\delta) + D(\delta)$ as in (26). Since $A(\delta) \geq 0$ and $\rho'(t) > -1$ for all t, we obtain

$$D(\delta) = \frac{\lambda}{\tau}\Big(p(\delta)\mathbb{E}[\rho'(\eta)] + (1 - p(\delta))\mathbb{E}[\rho'(-\eta)]\Big) > -\frac{\lambda}{\tau}.$$

Hence $\mathcal{B}'(\delta) > -\lambda/\tau$, and therefore

$$M'(\delta) = \lambda + \frac{\sigma_x^2}{\tau}\mathcal{B}'(\delta) > \lambda - \frac{\sigma_x^2}{\tau}\frac{\lambda}{\tau} = \lambda\left(1 - \frac{\sigma_x^2}{\tau^2}\right).$$

Since $\tau^2 \geq \sigma_x^2$ and $\lambda > 0$ by definition, $M'(\delta) > 0$ for all δ, so M is strictly increasing in δ (equivalently in u_x).

Undersensitivity away from the reference point. Fix parameters and let $|\delta| \to \infty$. Note that $p'(\delta) = \phi(\delta/\sigma_o)/\sigma_o \to 0$ and $p(\delta) \to 1$ as $\delta \to +\infty$ (resp. $p(\delta) \to 0$ as $\delta \to -\infty$). Also $\eta = \lambda(\delta + \epsilon)/\tau \to +\infty$ a.s. as $\delta \to +\infty$ (resp. $\eta \to -\infty$ a.s. as $\delta \to -\infty$). Since $\rho(t) \to 0$ as $t \to +\infty$ and $\rho'(t) \to 0$ as $t \to +\infty$, dominated convergence yields that $\mathbb{E}[\rho(\eta)] \to 0$ and $\mathbb{E}[\rho'(\eta)] \to 0$ as $\delta \to +\infty$. For the terms involving $-\eta$, note that $|\rho'(\cdot)| < 1$, so

$$\Big|(1 - p(\delta))\mathbb{E}[\rho'(-\eta)]\Big| \leq 1 - p(\delta) \to 0.$$

Moreover, for $x > 0$ we have $\rho(-x) \leq x + 1/x$ (Claim 1), so $\rho(-\eta) \leq |\eta| + 1$ on $\{|\eta| \geq 1\}$ and $\rho(-\eta) \leq \rho(-1)$ on $\{|\eta| < 1\}$. Hence there is a finite constant K (depending only on primitives) such that $\mathbb{E}[\rho(-\eta)] \leq K + \mathbb{E}|\eta|$, and $\mathbb{E}|\eta| = O(|\delta|)$. Since $1 - p(\delta) = 1 - \Phi(\delta/\sigma_o)$ decays at a Gaussian rate in δ while $\mathbb{E}|\eta|$ grows at most linearly, it follows that $(1-p(\delta))\mathbb{E}[\rho(-\eta)] \to 0$ as $\delta \to +\infty$.

Combining these limits in (24)–(26) gives $\mathcal{B}(\delta) \to 0$ and $\mathcal{B}'(\delta) \to 0$ as $|\delta| \to \infty$. Therefore $M'(\delta) \to \lambda \in (0, 1)$ as $|\delta| \to \infty$, so there exists $\varepsilon' > 0$ such that $M'(\delta) < 1$ whenever $|\delta| > \varepsilon'$.

Part 1(a): oversensitivity near the reference point when σ_o is small. At $\delta = 0$ we have $p(0) = 1/2$, $p'(0) = \phi(0)/\sigma_o$, and $\eta = \lambda\epsilon/\tau$ has mean 0. Using (26) and symmetry,

$$B'(0) = p'(0)\Big(\mathbb{E}[\rho(\eta)] + \mathbb{E}[\rho(-\eta)]\Big) + \frac{\lambda}{\tau}\mathbb{E}[\rho'(\eta)] = \frac{\sqrt{2/\pi}}{\sigma_o}\mathbb{E}[\rho(\eta)] + \frac{\lambda}{\tau}\mathbb{E}[\rho'(\eta)].$$

By convexity of ρ (Claim 1) and $\mathbb{E}[\eta] = 0$, Jensen implies $\mathbb{E}[\rho(\eta)] \geq \rho(0) = \sqrt{2/\pi}$. Also $-1 < \rho'(t) < 0$ (Claim 1), hence $\mathbb{E}[\rho'(\eta)] > -1$. Therefore

$$B'(0) \;\geq\; \frac{\sqrt{2/\pi}}{\sigma_o}\sqrt{\frac{2}{\pi}} - \frac{\lambda}{\tau} = \frac{2}{\pi\sigma_o} - \frac{\lambda}{\tau},$$

and so

$$M'(0) = \lambda + \frac{\sigma_x^2}{\tau} B'(0) \xrightarrow[\sigma_o \downarrow 0]{} \infty,$$

because $\tau \to \sqrt{\sigma_x^2 + \sigma_r^2} > 0$ as $\sigma_o \downarrow 0$. Hence there exists $\bar{\sigma} > 0$ such that $M'(0) > 1$ for all $0 < \sigma_o < \bar{\sigma}$. (The existence of such $\bar{\sigma} > 0$ is all that we need here; later, we will define $\bar{\sigma} := \inf\{\sigma > 0 : \sup_\delta M'(\delta) \leq 1 \ \forall \sigma_0 \geq \sigma\}$, and provide arguments to show that this is bounded.) By continuity of $M'(\cdot)$ in δ, for each such σ_o there exists $\varepsilon > 0$ with $M'(\delta) > 1$ whenever $|\delta| < \varepsilon$.

Part 1(b): only undersensitivity when $\sigma_o \geq \bar{\sigma}$. From (27)–(26) and Claim 1,

$$|D(\delta)| \leq \frac{\lambda}{\tau} \qquad \text{for all } \delta.$$

For $A(\delta)$, use the bound $\rho(t) \leq K_0 + |t|$ valid for all t (e.g. take $K_0 := \max\{\rho(0), \rho(-1)\}$ and use Claim 1(i),(iv) piecewise), which implies $\mathbb{E}[\rho(\pm\eta)] \leq K_0 + \mathbb{E}|\eta|$ and $\mathbb{E}|\eta| \leq |m| + v\sqrt{2/\pi}$ with $m = \lambda\delta/\tau$ and $v = \lambda\sigma_\epsilon/\tau$. Writing $t := \delta/\sigma_o$ so that $p'(\delta) = (1/\sigma_o)\phi(t)$, we obtain

$$A(\delta) = \frac{1}{\sigma_o}\phi(t)\Big(\mathbb{E}[\rho(\eta)] + \mathbb{E}[\rho(-\eta)]\Big) \leq \frac{1}{\sigma_o}\phi(t)\Big(2K_0 + 2|m| + 2v\sqrt{2/\pi}\Big).$$

Since $\sup_{t\in\mathbb{R}} \phi(t) = \phi(0)$ and $\sup_{t\in\mathbb{R}} |t|\phi(t) = \phi(1)$, the right-hand side is bounded by

$$\sup_\delta A(\delta) \ \leq \ \frac{K_1}{\sigma_o} + \frac{K_2}{\tau}$$

for finite constants K_1, K_2 depending only on $(\lambda, \sigma_\epsilon, K_0)$. Using $\tau \geq \sigma_o$, it follows that $\sup_\delta |B'(\delta)| \leq K/\sigma_o$ for some finite K, and hence

$$\sup_\delta \Big| M'(\delta) - \lambda \Big| \leq \frac{\sigma_x^2}{\tau} \sup_\delta |B'(\delta)| \leq \frac{K\sigma_x^2}{\sigma_o^2}.$$

Because $\lambda \in (0,1)$, choose $\bar{\sigma}$ large enough so that $\lambda + K\sigma_x^2/\bar{\sigma}^2 < 1$; in fact, let $\bar{\sigma} := \inf\{\sigma > 0 : \sup_\delta M'(\delta) \leq 1 \ \forall \sigma_0 \geq \sigma\}$; the arguments above show it is bounded. (Note that such $\bar{\sigma}$ works for both Part 1(a) and (b).) Then for every $\sigma_o \geq \bar{\sigma}$ we have $M'(\delta) < 1$ for all δ. Together with strict monotonicity (so $M'(\delta) > 0$), this yields $0 < M'(\delta) < 1$ for all δ.

Part 1(c): noiseless ordinal signal ($\sigma_o = 0$). When $\sigma_o = 0$, the ordinal signal is deterministic: $o = +$ if and only if $\delta > 0$, and $o = -$ if $\delta < 0$. At $\delta = 0$, the ordinal signal switches discretely, creating a discontinuous jump in M: for $\delta \downarrow 0$, the DM receives $o = +$ and the boost is positive; for $\delta \uparrow 0$, the DM receives $o = -$ and the boost is negative. For $\delta \neq 0$, the ordinal signal is fixed, and $M(\delta)$ is smooth with

$$M'(\delta) = \lambda + \frac{\sigma_x^2}{\tau} \cdot \frac{\lambda}{\tau}\,\mathbb{E}[\rho'(\eta)] < \lambda < 1,$$

since $A(\delta) = 0$ (no stochastic switching) and $\rho' < 0$.

Part 2: Over- and under-estimation. Suppose first that $0 < \sigma_o < \bar{\sigma}$. By the symmetry of the model under $\tilde{u}_x = \tilde{u}_r = u_r$, $M(0) = u_r = u_x$, so there is no bias at $\delta = 0$. Define $\beta(\delta) := M(\delta) - u_x = (\lambda - 1)\delta + (\sigma_x^2/\tau)\,\mathcal{B}(\delta)$. We begin studying behavior near $\delta = 0$. We have $\beta(0) = 0$ and

$$\beta'(\delta) = M'(\delta) - 1.$$

Since $M'(0) > 1$ by Part 1(a), $\beta'(0) > 0$. Hence $\beta(\delta) > 0$ for small $\delta > 0$ and $\beta(\delta) < 0$ for small $\delta < 0$. Finally, symmetry of the Gaussian environment implies $M(-\delta) - u_r = -(M(\delta) - u_r)$ for all δ, and therefore $\beta(-\delta) = -\beta(\delta)$. This implies that, when $|\delta|$ is small, we have: if $\delta > 0$, then $\beta(\delta) > 0$ thus $M(\delta) > u_x$, giving us overestimation; $\delta < 0$, then $\beta(\delta) < 0$ thus $M(\delta) < u_x$, underestimation, as sought.

Suppose $\delta > 0$. u_x (and δ) increases, the average $\tilde{u}_x^s$ increases only by λ, and the boost is strictly increasing first and then strictly decreasing, as shown in Proposition 15. Define $\Delta^* > 0$ as the unique maximizer of $C(\delta)$ given by Proposition 15(c)(ii). To rigorously establish that $\beta(\delta)$ crosses zero exactly once, we write $\beta(\delta) = \mathbb{E}_\delta\left[B(z_s) - \frac{(1-\lambda)\tau}{\lambda} z_s\right]$. Observe that for $z > 0$, the function $f(z) := B(z) - \frac{(1-\lambda)\tau}{\lambda} z$ exhibits a single-crossing property from positive to negative, because $B(z)$ is bounded and decaying as $z \to \infty$ while the linear term grows unbounded, and f is strictly decreasing (since $f'(z) = B'(z) - c$, where $B'(z) = p\rho'(z) + (1-p)\rho'(-z)$; since $\rho'(t) \in (-1, 0)$ for all t by Claim 1, we have $B'(z) \in (-1, 0)$, and therefore $f'(z) < -c < 0$). Since the family of normal distributions $z_s \sim \mathcal{N}(\lambda\delta/\tau, \sigma_z^2)$ satisfies the strict Monotone Likelihood Ratio Property (MLRP) with respect to δ, Karlin's theorem on Total Positivity guarantees that the expectation $\mathbb{E}_\delta[f(z_s)]$ crosses zero exactly once from positive to negative for $\delta > 0$. With $\beta(0) = 0$ and $\beta'(0) > 0$, this guarantees strict overestimation when δ is positive and below a threshold, and underestimation after it.

With the result above, this implies overestimation when δ is positive and below a threshold, and underestimation after it. The symmetric results for negative δ follow by the same arguments.

When $\sigma_o \geq \bar{\sigma}$, $M'(\delta) < 1$ everywhere (Part 1(b)), so $\beta'(\delta) < 0$ for all δ. Since $\beta(0) = 0$, this gives $\beta(\delta) < 0$ for $\delta > 0$ and $\beta(\delta) > 0$ for $\delta < 0$: pure under-/over-estimation, as stated. *Part*

3: Effects of uncertainty. Write

$$M(\delta) = u_r + \lambda\delta + C(\delta), \qquad C(\delta) := \frac{\sigma_x^2}{\tau} B(\delta),$$

where $B(\delta)$ is the average boost from the ordinal signal. Fix $\delta > 0$.

First, we note the effect of σ_x^2 holding λ fixed. The term $u_r + \lambda\delta$ does not vary with σ_x^2 when λ is held fixed, so

$$\frac{\partial M(\delta)}{\partial \sigma_x^2} = \frac{\partial C(\delta)}{\partial \sigma_x^2}.$$

By Proposition 15(a),

$$\frac{\partial C(\delta)}{\partial \sigma_x^2} > 0 \qquad \text{for } \delta > 0.$$

Hence

$$\frac{\partial M(\delta)}{\partial \sigma_x^2} > 0.$$

(a) Effect of σ_r^2. Since λ and σ_x^2 do not depend on σ_r^2,

$$\frac{\partial M(\delta)}{\partial \sigma_r^2} = \frac{\partial C(\delta)}{\partial \sigma_r^2}.$$

Proposition 15(b) yields a cutoff $\Delta_r > 0$ such that

$$\frac{\partial M(\delta)}{\partial \sigma_r^2} < 0 \quad \text{for } 0 < \delta < \Delta_r, \qquad \frac{\partial M(\delta)}{\partial \sigma_r^2} > 0 \quad \text{for } \delta > \Delta_r.$$

(b) Effect of σ_ϵ^2. Holding σ_p^2 fixed, $\partial\lambda/\partial\sigma_\epsilon^2 < 0$ and $\partial\sigma_x^2/\partial\sigma_\epsilon^2 > 0$. Since $C(\delta)$ depends on $(\sigma_p^2, \sigma_\epsilon^2)$ through both σ_x^2 and λ,

$$\frac{\partial M(\delta)}{\partial \sigma_\epsilon^2} = \frac{\partial C(\delta)}{\partial \sigma_x^2}\frac{\partial \sigma_x^2}{\partial \sigma_\epsilon^2} + \big(\delta + C_\lambda(\delta)\big)\frac{\partial \lambda}{\partial \sigma_\epsilon^2}.$$

By Proposition 15(a), the first term is positive. By Proposition 15(c), $C(\delta) \to 0$ as $\delta \to \infty$. By dominated convergence, $C_{\sigma_x^2}(\delta) \to 0$ and $C_\lambda(\delta) \to 0$ as $\delta \to \infty$. Therefore

$$\frac{\partial M(\delta)}{\partial \sigma_\epsilon^2} \sim \delta \frac{\partial \lambda}{\partial \sigma_\epsilon^2} < 0 \qquad \text{as } \delta \to \infty.$$

Hence there exists $\Delta_\epsilon \geq 0$ such that

$$\frac{\partial M(\delta)}{\partial \sigma_\epsilon^2} < 0 \qquad \text{for all } \delta > \Delta_\epsilon.$$

Finally, consider the effect of σ_p^2. Hold σ_ϵ^2 fixed and write

$$e := \sigma_\epsilon^2, \qquad c := \sigma_r^2 + \sigma_o^2, \qquad a := \sigma_x^2 = \frac{\sigma_p^2 \sigma_\epsilon^2}{\sigma_p^2 + \sigma_\epsilon^2} = \lambda e, \qquad \tau := \sqrt{a + c}.$$

Since $\lambda = \sigma_p^2/(\sigma_p^2 + \sigma_\epsilon^2)$, it is enough to prove that

$$\frac{\partial M(\delta)}{\partial \lambda} > 0,$$

because

$$\frac{\partial \lambda}{\partial \sigma_p^2} = \frac{\sigma_\epsilon^2}{(\sigma_p^2 + \sigma_\epsilon^2)^2} > 0.$$

Let $y := \delta + \epsilon$, so $y \sim N(\delta, \sigma_\epsilon^2)$, and define

$$p := \Pr(o = + \mid u_x, u_r) = \begin{cases} 1, & \sigma_o^2 = 0, \\ \Phi(\delta/\sigma_o), & \sigma_o^2 > 0, \end{cases} \qquad H_p(z) := p\rho(z) - (1-p)\rho(-z).$$

Using (24)–(25), we can write

$$M(\delta) - u_r = E_\epsilon\left[\lambda y + \frac{a}{\tau} H_p\left(\frac{\lambda y}{\tau}\right)\right]. \tag{A.4}$$

Define

$$z := \frac{\lambda y}{\tau}.$$

Differentiating the integrand in (A.4) with respect to λ, using $a = \lambda e$, yields

$$\frac{\partial}{\partial \lambda}\left[\lambda y + \frac{a}{\tau} H_p(z)\right] = \frac{\tau}{\lambda}\, q_{\alpha,p}(z), \tag{A.5}$$

where

$$\alpha := \frac{a(c + a/2)}{\tau^4} = \frac{1}{2}\left(1 - \frac{c^2}{\tau^4}\right) \in (0, \tfrac{1}{2}],$$

and $q_{\alpha,p}$ is defined in Claim 2.

Therefore,

$$\frac{\partial M(\delta)}{\partial \lambda} = \frac{\tau}{\lambda} E[q_{\alpha,p}(Z)], \tag{A.6}$$

where

$$Z := \frac{\lambda(\delta + \epsilon)}{\tau} \sim N\left(\frac{\lambda \delta}{\tau}, \frac{\lambda^2 \sigma_\epsilon^2}{\tau^2}\right).$$

If $\delta > 0$, then $E[Z] = \lambda\delta/\tau > 0$, and also $p \geq 1/2$, with strict inequality if $\sigma_o^2 > 0$ (and $p = 1$ if $\sigma_o^2 = 0$). By Claim 2,

$$E[q_{\alpha,p}(Z)] > 0.$$

Hence, from (A.6),

$$\frac{\partial M(\delta)}{\partial \lambda} > 0.$$

Since $\partial\lambda/\partial\sigma_p^2 > 0$, we conclude

$$\frac{\partial M(\delta)}{\partial \sigma_p^2} > 0 \qquad \text{for } \delta > 0.$$

The case $\delta < 0$ follows by symmetry. □

D.3 Proof of Proposition 3

Fix primitives $(\tilde{u}_x, \tilde{u}_r, \sigma_x^2, \sigma_r^2, \sigma_o^2)$ and let

$$d \;:=\; \sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}, \qquad w \;:=\; \frac{\sigma_x^2}{d}, \qquad z_s \;:=\; \frac{\tilde{u}_x^s - \tilde{u}_r}{d}.$$

By Proposition 1, for each realization $(s, o) \in \mathbb{R} \times \{+, -\}$,

$$\tilde{u}_x^{s,o} \;=\; \mathbb{E}[u_x \mid s, o] \;=\; \tilde{u}_x^s \;+\; w\,\psi(z_s, o), \tag{28}$$

where $\psi(z,+) = \phi(z)/\Phi(z)$ and $\psi(z,-) = -\phi(-z)/\Phi(-z)$. Moreover, conditional on the true (u_x, u_r),

$$p(u_x, u_r) \;:=\; \Pr(o = + \mid u_x, u_r) = \begin{cases} \Phi\!\left(\frac{u_x - u_r}{\sigma_o}\right), & \sigma_o > 0, \\ \mathbf{1}\{u_x \geq u_r\}, & \sigma_o = 0. \end{cases}$$

Since $p(u_x, u_r)$ does not depend on s, we can write the mean posterior as

$$\mathbb{E}[\tilde{u}_x^{s,o}] \;=\; \mathbb{E}_s\big[p\,\tilde{u}_x^{s,+} + (1-p)\,\tilde{u}_x^{s,-}\big]. \tag{29}$$

Part 1. Effect of u_r (holding $\tilde{u}_r$ fixed). Differentiating (29) w.r.t. u_r (holding $\tilde{u}_r$ fixed) yields

$$\frac{\partial}{\partial u_r}\mathbb{E}[\tilde{u}_x^{s,o}] \;=\; \frac{\partial p}{\partial u_r}\,\mathbb{E}_s\big[\tilde{u}_x^{s,+} - \tilde{u}_x^{s,-}\big].$$

Using (28),

$$\tilde{u}_x^{s,+} - \tilde{u}_x^{s,-} = w\big(\psi(z_s,+) - \psi(z_s,-)\big) = w\left(\frac{\phi(z_s)}{\Phi(z_s)} + \frac{\phi(z_s)}{1-\Phi(z_s)}\right) = w\,\frac{\phi(z_s)}{\Phi(z_s)(1-\Phi(z_s))} \;>\; 0.$$

Since $\sigma_o > 0$, then $\partial p/\partial u_r = -\phi((u_x - u_r)/\sigma_o)/\sigma_o < 0$, hence $\partial\mathbb{E}[\tilde{u}_x^{s,o}]/\partial u_r < 0$.[45] This proves Part 1.

Part 2. Effect of $\tilde{u}_r$ (holding u_r fixed). Since p does not depend on $\tilde{u}_r$, differentiating (28) gives

$$\frac{\partial}{\partial \tilde{u}_r}\tilde{u}_x^{s,o} \;=\; w\,\psi_z(z_s,o)\cdot\frac{\partial z_s}{\partial \tilde{u}_r} \;=\; -\frac{w}{d}\,\psi_z(z_s,o),$$

where ψ_z denotes the derivative w.r.t. the first argument. It is well-known (and easily verified) that the inverse Mills ratio $m(z) := \phi(z)/\Phi(z)$ satisfies $m'(z) = -m(z)\big(z + m(z)\big) < 0$ for all z; hence $\psi_z(z,+) = m'(z) < 0$. Moreover $\psi(z,-) = -m(-z)$ implies $\psi_z(z,-) = m'(-z) < 0$ as well. Therefore $-\psi_z(z_s,o) > 0$ for both $o \in \{+,-\}$, and thus $\partial\tilde{u}_x^{s,o}/\partial\tilde{u}_r > 0$ pointwise. Taking expectations over (s,o) yields $\partial\mathbb{E}[\tilde{u}_x^{s,o}]/\partial\tilde{u}_r > 0$. This proves Part 2.

Part 3. When $\tilde{u}_r = \alpha u_r + (1-\alpha)\bar{u}_r$. In this case, the total derivative becomes

$$\frac{d}{du_r}\mathbb{E}[\tilde{u}_x^{s,o}] \;=\; \frac{\partial}{\partial u_r}\mathbb{E}[\tilde{u}_x^{s,o}] \;+\; \frac{\partial}{\partial \tilde{u}_r}\mathbb{E}[\tilde{u}_x^{s,o}]\cdot\frac{d\tilde{u}_r}{du_r} \;=\; \frac{\partial}{\partial u_r}\mathbb{E}[\tilde{u}_x^{s,o}] \;+\; \alpha\,\frac{\partial}{\partial \tilde{u}_r}\mathbb{E}[\tilde{u}_x^{s,o}].$$

By Parts 1–2, $\partial\mathbb{E}[\tilde{u}_x^{s,o}]/\partial u_r < 0$ and $\partial\mathbb{E}[\tilde{u}_x^{s,o}]/\partial\tilde{u}_r > 0$. It follows that we have

$$\frac{d}{du_r}\mathbb{E}[\tilde{u}_x^{s,o}] > 0 \quad\Longleftrightarrow\quad \alpha \;>\; \alpha^\star \;:=\; -\,\frac{\frac{\partial}{\partial u_r}\mathbb{E}[\tilde{u}_x^{s,o}]}{\frac{\partial}{\partial \tilde{u}_r}\mathbb{E}[\tilde{u}_x^{s,o}]},$$

and $\frac{d}{du_r}\mathbb{E}[\tilde{u}_x^{s,o}] < 0$ iff $\alpha < \alpha^\star$. □

D.4 Proof of Proposition 4

Let

$$\mu(w) := a^{\text{no rp}}(w) \;=\; \lambda\beta w + (1-\lambda)\,\overline{\beta w}, \qquad t(a,w) := \frac{\mu(w)a + \lambda\epsilon - \beta r}{\sigma_x}.$$

Then, if we define

$$f(a;w) \;:=\; \mu(w)\,a - \frac{a^2}{2} \;+\; \sigma_x\,\psi\big(t(a,w),o_a\big) \;+\; \lambda\epsilon, \tag{30}$$

[45]If $\sigma_o = 0$, then $p(u_x,u_r) = \mathbf{1}\{u_x \geq u_r\}$ is locally constant in u_r as long as $u_x \neq u_r$, so $\partial\mathbb{E}[\tilde{u}_x^{s,o}]/\partial u_r = 0$ as long as $u_x \neq u_r$. When u_r surpasses u_x, instead, $p(u_x,u_r)$ jumps from 0 to 1.

our maximization problem is equivalent to $\max_{a\geq 0} f(a,w)$.

Step 1: Basic Facts We begin by establishing a few basic facts. Let

$$m(t) := \frac{\phi(t)}{\Phi(t)}, \qquad n(t) := \frac{\phi(t)}{1-\Phi(t)}.$$

Then, a well-known property of Mill's ratios (easily verified) is that

$$m'(t) = -m(t)\big(t+m(t)\big) \in (-1,0), \qquad m''(t) > 0,$$

$$n'(t) = n(t)\big(n(t)-t\big) \in (0,1), \qquad n''(t) > 0,$$

and by Normal symmetry $m(t) = n(-t)$ and $n'(t) = -m'(-t)$.

Notice that for $a \neq r/w$ we have

$$\frac{\partial t}{\partial a} = \frac{\mu(w)}{\sigma_x}, \qquad f_a(a;w) = \mu(w) - a + \mu(w)\,\psi'\big(t(a,w), o_a\big).$$

Thus, the left and right derivatives at $a = r/w$ are

$$\begin{aligned} D_-(w) &:= \lim_{a\uparrow r/w} f_a(a;w) = \mu(w) - \frac{r}{w} - \mu(w)\,n'\big(t(r/w,w)\big),\\ D_+(w) &:= \lim_{a\downarrow r/w} f_a(a;w) = \mu(w) - \frac{r}{w} + \mu(w)\,m'\big(t(r/w,w)\big). \end{aligned} \tag{31}$$

Given the properties above, $D_\pm(\cdot)$ are continuous in w, with

$$\lim_{w\downarrow 0} D_\pm(w) = -\infty, \qquad \lim_{w\to\infty} D_\pm(w) = +\infty.$$

Hence there exist $w_-, w_+ > 0$ such that $D_-(w_-) = 0$ and $D_+(w_+) = 0$. Moreover, using $m(t) = n(-t)$ and $m'' > 0$ one checks that $D_-(w) - D_+(w)$ is strictly increasing in w, giving $w_- < w_+$.

Next, we establish the concavity of the problems on the left and right. For $a < r/w$,

$$f_{aa}(a;w) = -1 - \frac{\mu(w)^2}{\sigma_x}\,n''\big(t(a,w)\big) < -1,$$

so $f(\cdot;w)$ is strictly concave on $(-\infty, r/w)$ and $f_a(\cdot;w)$ is strictly decreasing there. For $a > r/w$,

$$f_{aa}(a;w) = -1 + \frac{\mu(w)^2}{\sigma_x}\,m''\big(t(a,w)\big),$$

so interior right-side stationary points that satisfy the second-order condition (SOC)

$$-1 + \frac{\mu(w)^2}{\sigma_x} m''(t) < 0$$

are strict local maxima.

Finally, notice that we have a positive jump of the objective function at the point of discontinuity: because $\psi(t,-) = -n(t)$ and $\psi(t,+) = m(t)$,

$$f\big(\tfrac{r}{w}^+; w\big) - f\big(\tfrac{r}{w}^-; w\big) = \sigma_x\big(m(t(r/w,w)) + n(t(r/w,w))\big) \;>\; 0.$$

Step 2. Solving the Left and Right Problems. For each w, consider the left problem $\max_{a \le r/w} f(a; w)$ and the right problem $\max_{a \ge r/w} f(a; w)$. On the left, strict concavity implies a unique maximizer $a^-(w)$: either $a^-(w) = r/w$ if $D_-(w) \ge 0$, or else $a^-(w) \in (0, r/w)$ solves

$$0 = \mu(w) - a - \mu(w)\, n'\big(t(a,w)\big).$$

On the right, any interior maximizer $a^+(w) > r/w$ satisfies

$$0 = \mu(w) - a + \mu(w)\, m'\big(t(a,w)\big), \qquad -1 + \frac{\mu(w)^2}{\sigma_x} m''\big(t(a,w)\big) < 0.$$

If instead $D_+(w) \le 0$, the right-side maximum is attained at the boundary $a = r/w$.

Let $V^-(w)$ and $V^+(w)$ denote the left- and right-side value functions. Because $f(\frac{r}{w}^+; w) - f(\frac{r}{w}^-; w) > 0$, evaluating at the discontinuity $a = \frac{r}{w}$ from the right weakly dominates evaluating from the left. Hence

$$V^+(w) \;\ge\; f\big(\tfrac{r}{w}^+; w\big) \;=\; f\big(\tfrac{r}{w}^-; w\big) + \sigma_x\big(m(t(r/w,w)) + n(t(r/w,w))\big) \;>\; f\big(\tfrac{r}{w}^-; w\big).$$

Step 3: Existence of w_1 and w_3 and bunching on (w_1, w_3). Consider the difference

$$\Delta_1(w) := V^+(w) - V^-(w).$$

As $w \downarrow 0$, $r/w \to \infty$ and (by the quadratic term) $f(\frac{r}{w}^+; w) \to -\infty$, hence $\Delta_1(w) < 0$ for small w. As $w \to \infty$, $\mu(w) \sim \lambda\beta w \to \infty$ and the right side dominates, so $\Delta_1(w) > 0$ for large w. By continuity, there exists $w_1 > 0$ with $\Delta_1(w_1) = 0$ and $\Delta_1(w) > 0$ for $w > w_1$ in a neighborhood of w_1.

Independently, by (31) and the limits above there exists $w_3 > w_1$ with $D_+(w_3) = 0$ and $D_+(w) < 0$ for $w < w_3$. Therefore, for every $w \in (w_1, w_3)$: (i) the right-side problem attains its maximum at the boundary $a = r/w$ (since it is decreasing just to the right of the boundary), and (ii) $\Delta_1(w) \geq 0$ so the right-side value weakly exceeds the left-side value. Hence the global maximizer is

$$a^*(w) = \frac{r}{w} \qquad \text{for all } w \in (w_1, w_3),$$

and $a^{*\prime}(w) = -r/w^2 < 0$. This proves item (1).

Step 4: Bunching of earnings. On (w_1, w_3) we have $a^*(w)w = r$, i.e. earnings are constant. At w_1 and w_3 there are upward jumps: at w_1 the optimizer switches from an interior left solution $a^-(w) < r/w$ to $a = \frac{r}{w}$, and at w_3 the right derivative D_+ turns nonnegative so the optimizer moves to an interior right solution $a^+(w) > r/w$. For $w < w_1$ or $w > w_3$, the solution is interior and satisfies the respective FOCs. Differentiating the FOC implicitly yields (using $\mu'(w) = \lambda\beta > 0$ and the properties of the Mills Ratio discussed in Step 1)

$$a'_w = \mu'(1 + \psi') \, \frac{1 + \frac{\mu^2}{\sigma_x}\psi''}{1 - \frac{\mu^2}{\sigma_x}\psi''},$$

where $\psi' = -n' \in (-1, 0)$ and $\psi'' = -n'' < 0$ on the left, and $\psi' = m' \in (-1, 0)$, $\psi'' = m'' > 0$ on the right (with SOC ensuring the denominator > 0 on the right). In both cases $a'_w > 0$, hence

$$\frac{d}{dw}\big(a^*(w)\, w\big) = a^*(w) + w\, a'_w \;>\; 0$$

outside (w_1, w_3). This proves item (2).

Step 4: Comparison with the reference-free benchmark and existence of w_2. For $w < w_1$ (left interior), the FOC gives $a^*(w) = \mu(w)\big(1 - n'(t)\big) < \mu(w) = a^{\text{no rp}}(w)$. For $w > w_3$ (right interior with SOC), the FOC gives $a^*(w) = \mu(w)\big(1 + m'(t)\big) < \mu(w) = a^{\text{no rp}}(w)$. For $w \in (w_1, w_3)$ we have $a^*(w) = r/w$; since $w \mapsto r/w$ is strictly decreasing and $w \mapsto \mu(w)$ strictly increasing, there exists a unique $w_2 > 0$ solving $r/w_2 = \mu(w_2)$, namely

$$w_2 := \frac{-\,(1-\lambda)\,\overline{\beta w} + \sqrt{(1-\lambda)^2\overline{\beta w}^{\,2} + 4\lambda\beta r}}{2\lambda\beta}.$$

At $w = w_3$, $D_+(w_3) = 0$ implies $\frac{r}{w_3} = \mu(w_3)\big(1 + m'(t(r/w_3, w_3))\big) < \mu(w_3)$, hence $w_2 < w_3$. At $w = w_1$, by the definition of w_1 via the value crossing and the monotonicity of Δ_1, one has $\frac{r}{w_1} > \mu(w_1)$, hence $w_1 < w_2$. Therefore $w_1 < w_2 < w_3$, and the inequalities in item (3) follow. $\square$

D.5 Proof of Lemma 1

Note that for any information set $\mathcal{I}$, we have $\min_a \mathbb{E}[(u_x - a)^2|\mathcal{I}] = \mathbb{V}[u_x|\mathcal{I}]$. Then $V(r)$ can be rewritten as the expected reduction in the posterior variance of u_x provided by the ordinal signal o, that is,

$$V(r) = \mathbb{E}_s\left[\mathbb{V}[u_x|s] - \mathbb{E}_{o|s}\left[\mathbb{V}[u_x|s,o]\right]\right],$$

where s denotes the cardinal signals s_x, s_r. Recall the Law of Total Variance, which specifies that for any random variables X, Y, $\mathbb{V}[Y] = \mathbb{E}[\mathbb{V}[Y|X]] + \mathbb{V}[\mathbb{E}[Y|X]]$, thus $\mathbb{V}[Y] - \mathbb{E}[\mathbb{V}[Y|X]] = \mathbb{V}[\mathbb{E}[Y|X]]$. Applying it to $Y = u_x$ and $X = o$, we obtain

$$V(r) = \mathbb{E}_s\left[\mathbb{V}_{o|s}\left[\mathbb{E}[u_x \mid s, o]\right]\right].$$

Let $z_s := \frac{\tilde{u}_x^s - \tilde{u}_r^s}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}$ and note that, conditional on s, the ordinal signal o is a binary random variable with $o = +$ with probability $p := P(u_x - u_r + \epsilon_o \geq 0 \mid s) = \Phi(z_s)$; and $o = -$ with probability $1 - p = 1 - \Phi(z_s)$.

For each s, denote by M the update to the posterior mean given by the ordinal signal, that is, $M := \mathbb{E}[u_x \mid s, o] - \tilde{u}_x^s$. Note that $\mathbb{V}_{o|s}[\mathbb{E}[u_x|s,o]] = \mathbb{V}_{o|s}[\tilde{u}_x^s + M] = \mathbb{V}_{o|s}[M|s]$, since $\tilde{u}_x^s$ is constant conditional on s. It follows that we have $V(r) = \mathbb{E}_s\left[\mathbb{V}_{o|s}[M|s]\right]$.

From Proposition 1, M is a binary variable that takes values:

$$m_+ = \frac{\sigma_x^2}{\sqrt{\Sigma}}\frac{\phi(z_s)}{\Phi(z_s)} \quad \text{and} \quad m_- = -\frac{\sigma_x^2}{\sqrt{\Sigma}}\frac{\phi(z_s)}{1-\Phi(z_s)}.$$

Moreover, note how we must have $\mathbb{E}[M \mid s] = 0$.[46] Since M is a binary variable with mean zero, its variance is the expected squared value $\mathbb{E}[M^2 \mid s]$. This means

$$\begin{aligned}
\mathbb{V}_{o|s}[M|s] &= p \cdot (m_+)^2 + (1-p) \cdot (m_-)^2 \\
&= \Phi(z_s)\left[\frac{\sigma_x^2}{\sqrt{\Sigma}}\frac{\phi(z_s)}{\Phi(z_s)}\right]^2 + (1-\Phi(z_s))\left[-\frac{\sigma_x^2}{\sqrt{\Sigma}}\frac{\phi(z_s)}{1-\Phi(z_s)}\right]^2 \\
&= \frac{\sigma_x^4}{\Sigma}\phi(z_s)^2\left[\frac{\Phi(z_s)}{\Phi(z_s)^2} + \frac{1-\Phi(z_s)}{(1-\Phi(z_s))^2}\right] \\
&= \frac{\sigma_x^4}{\Sigma}\phi(z_s)^2\left[\frac{1}{\Phi(z_s)} + \frac{1}{1-\Phi(z_s)}\right] = \frac{\sigma_x^4}{\Sigma}\left[\frac{\phi(z_s)^2}{\Phi(z_s)(1-\Phi(z_s))}\right] = \frac{\sigma_x^4}{\Sigma}h(z_s).
\end{aligned}$$

[46] To see why, $\mathbb{E}[M \mid s] = p \cdot m_+ + (1-p) \cdot m_- = \frac{\sigma_x^2}{\sqrt{\Sigma}}\left[\Phi(z_s)\frac{\phi(z_s)}{\Phi(z_s)} - (1-\Phi(z_s))\frac{\phi(z_s)}{1-\Phi(z_s)}\right] = 0$.

Since we proved $V(r) = \mathbb{E}_s\left[\mathbb{V}_{o|s}[M|s]\right]$, taking the expectation of the expression above over s completes the proof. $\square$

D.6 Proof of Proposition 5

In what follows, let $\Sigma := \sigma_x^2 + \sigma_r^2 + \sigma_o^2$ and $b = \tilde{u}_x - \tilde{u}_r$. Recall from Lemma 1 that

$$V(r) = \frac{\sigma_x^4}{\Sigma}\mathbb{E}_{s_x,s_r}\left[h\left(\frac{\tilde{u}_r^{s_r} - \tilde{u}_x^{s_x}}{\sqrt{\Sigma}}\right)\right].$$

Step 1: Monotonicity in Bias. Note that $\tilde{u}_x^s$ and $\tilde{u}_r^s$ are normally distributed random variables (the posterior means after receiving cardinal signals). The normalized difference $D_b = \frac{\tilde{u}_x^s - \tilde{u}_r^s}{\sqrt{\Sigma}}$ is therefore normally distributed with mean $\frac{b}{\sqrt{\Sigma}}$ and variance $\sigma_D^2 = \frac{\mathrm{Var}(\tilde{u}_x^s)+\mathrm{Var}(\tilde{u}_r^s)}{\Sigma}$.

Note how the function h is even.[47] Similarly, the distribution of D when the bias is b is the mirror image of the distribution when the bias is $-b$: D_b has the same distribution of $-D_{-b}$. It follows that

$$\mathbb{E}[h(D_b)] = \mathbb{E}[h(-D_{-b})] = \mathbb{E}[h(D_{-b})],$$

and thus $V(r)$ is an even function of b.

We are left with proving that V is strictly decreasing in $|b|$. Because h is even, differentiable, and strictly decreasing on $(0,\infty)$, its derivative h' is odd with $h'(t) < 0$ for $t > 0$ and $h'(t) > 0$ for $t < 0$.

Recall that we can write $D_b = \frac{b}{\sqrt{\Sigma}} + \sigma_D Z$ with $Z \sim \mathcal{N}(0,1)$. By differentiation under the integral sign (justified by dominated convergence)

$$\frac{\partial}{\partial b}\mathbb{E}[h(D_b)] = \mathbb{E}\left[h'(D_b)\,\frac{\partial D_b}{\partial b}\right] = \frac{1}{\sqrt{\Sigma}}\mathbb{E}[h'(D_b)].$$

For $b > 0$, the distribution of D_b places more mass on positive values than on negative values, and h' is negative on $(0,\infty)$ and positive on $(-\infty,0)$, so $\mathbb{E}[h'(D_b)] < 0$ for all $b > 0$, and strictly so because the distribution of D_b has full support.

Thus for $b > 0$, $\frac{\partial}{\partial b}\mathbb{E}[h(D_b)] < 0$, and, since $V(r)$ is a positive scalar multiple of $\mathbb{E}[h(D_b)]$, $\frac{\partial V}{\partial b}(b) < 0$ for $b > 0$. By evenness, V is strictly decreasing in $|b|$.

Step 2: Monotonicity in Signal Noise $\sigma_{\epsilon,r}^2$. Note that for any information structure $\mathcal{I}$, we have $\min_a \mathbb{E}[(u_x - a)^2|\mathcal{I}] = \mathbb{V}[u_x|\mathcal{I}]$. Then $V(r)$ can be rewritten as the expected reduction in the

[47] Indeed, $h(-t) = \frac{\phi(-t)^2}{\Phi(-t)(1-\Phi(-t))} = \frac{\phi(t)^2}{(1-\Phi(t))\Phi(t)} = h(t)$, using $\phi(-t) = \phi(t)$ and $\Phi(-t) = 1 - \Phi(t)$.

posterior variance of u_x provided by the ordinal signal o, that is,

$$V(r) = \mathbb{E}\left[\mathbb{V}[u_x|s]\right] - \mathbb{E}\left[\mathbb{V}[u_x|s,o]\right],$$

where s denotes the cardinal signals s_x, s_r. Recall the Law of Total Variance, which specifies that for any random variables X, Y, $\mathbb{V}[Y] = \mathbb{E}[\mathbb{V}[Y|X]] + \mathbb{V}[\mathbb{E}[Y|X]]$, thus $\mathbb{E}[\mathbb{V}[Y|X]] = \mathbb{V}[Y] - \mathbb{V}[\mathbb{E}[Y|X]]$. Applying it to the formulas above and simplifying the common term (the prior variance), yields

$$V(r) = \mathrm{Var}(\mathbb{E}[u_x \mid s, o]) - \mathrm{Var}(\mathbb{E}[u_x \mid s]).$$

Note that the second term does not vary in $\sigma^2_{\epsilon,r}$. Therefore, to show that that $V(r)$ decreases in $\sigma^2_{\epsilon,r}$ it is sufficient to show that $\mathrm{Var}(\mathbb{E}[u_x \mid s, o])$ decreases in $\sigma^2_{\epsilon,r}$.

Consider two noise levels $0 < \sigma^2_{\varepsilon,r,L} < \sigma^2_{\varepsilon,r,H}$. Write $s^L_r = u_r + \varepsilon^L_r$ with $\varepsilon^L_r \sim \mathcal{N}(0, \sigma^2_{\varepsilon,r,L})$ and $s^H_r = u_r + \varepsilon^H_r$ with $\varepsilon^H_r \sim \mathcal{N}(0, \sigma^2_{\varepsilon,r,H})$, and assume $\varepsilon^H_r = \varepsilon^L_r + \eta$ where $\eta \sim \mathcal{N}(0, \sigma^2_{\varepsilon,r,H} - \sigma^2_{\varepsilon,r,L})$ is independent of everything else. Let

$$\mathcal{I}_L := (s_x, s^L_r, o), \qquad \mathcal{I}_H := (s_x, s^H_r, o).$$

By construction, the conditional distribution of $\mathcal{I}_H$ given the state (u_x, u_r) is obtained from that of $\mathcal{I}_L$ by adding the independent noise η to s^L_r. Thus $\mathcal{I}_H$ is a garbling of $\mathcal{I}_L$ in the sense of Blackwell.

Blackwell's theorem then implies that $\mathcal{I}_L$ is more informative than $\mathcal{I}_H$ for any decision problem with state (u_x, u_r). In particular, consider the estimation problem for u_x under squared loss. For such a problem, a standard corollary of Blackwell dominance is that the more informative information structure yields a posterior mean with larger variance:

$$\mathbb{V}\big(\mathbb{E}[u_x \mid \mathcal{I}_L]\big) \;\geq\; \mathbb{V}\big(\mathbb{E}[u_x \mid \mathcal{I}_H]\big),$$

with strict inequality unless the two experiments are equivalent for this problem. This shows that $\mathrm{Var}(\mathbb{E}[u_x \mid s, o])$ is strictly decreasing in $\sigma^2_{\varepsilon,r}$, yielding the claim. □

D.7 Proof of Observation 1

Noting the assumption $\sigma^2_{\epsilon_x}, \sigma^2_{\epsilon_{r_\mathrm{A}}} \to \infty$, $\sigma^2_{p,r_\mathrm{SQ}} = 0$, and $\sigma^2_{o_\mathrm{SQ}} = \sigma^2_{o_\mathrm{A}} = 0$, Eq. 10 (derived from Lemma 1) implies

$$V(r_\mathrm{SQ}) > V(r_\mathrm{A}) \;\Leftrightarrow\; \frac{\sigma_x^4}{\sigma_x^2} h\left(\frac{\tilde{u}_{r_\mathrm{SQ}} - \tilde{u}_x}{\sqrt{\sigma_x^2}}\right) > \frac{\sigma_x^4}{\sigma_x^2 + \sigma^2_{r_\mathrm{A}}} h\left(\frac{\tilde{u}_r - \tilde{u}_x}{\sqrt{\sigma_x^2 + \sigma^2_{r_A}}}\right).$$

Noting also the assumption $b_{\mathrm{SQ}} := |\tilde{u}_{r_{\mathrm{SQ}}} - \tilde{u}_x| > 0$ and $\tilde{u}_{r_{\mathrm{A}}} = \tilde{u}_x$, and since $h(0) = \frac{2}{\pi}$

$$V(r_{\mathrm{SQ}}) > V(r_{\mathrm{A}}) \iff h\left(\frac{b_{\mathrm{SQ}}}{\sigma_x^2}\right) > \frac{2}{\pi}\frac{\sigma_x^2}{\sigma_x^2 + \sigma_{r_{\mathrm{A}}}^2}.$$

Since h is even and attains a maximum at 0, then

$$V(r_{\mathrm{SQ}}) > V(r_{\mathrm{A}}) \iff b_{\mathrm{SQ}} < \sigma_x^2 h^{-1}\left(\frac{2}{\pi}\frac{\sigma_x^2}{\sigma_x^2 + \sigma_{r_{\mathrm{A}}}^2}\right)$$

which concludes the proof. □

D.8 Proof of Proposition 6

Let $\sigma_T^2 := \sigma_x^2 + \sigma_r^2 + \sigma_o^2$, $d := \frac{\tilde{u}_r^{s_r} - \tilde{u}_x^{s_x}}{\sqrt{\sigma_T^2}}$, and $s := (s_x, s_r)$. Moreover, denote $V_s(r) := V(r \mid s_x, s_r)$ and recall from Lemma 1 that

$$V_s(r) = \frac{\sigma_x^4}{\sigma_T^2} h(d).$$

Note that, when we vary σ_r^2 or σ_o^2, for fixed (s_x, s_r), the posterior means $\tilde{u}_x^{s_x}, \tilde{u}_r^{s_r}$ and σ_x^2 are constant; only σ_T^2 changes. We can then differentiate V_s with respect to σ_T^2. By chain rule,

$$\frac{\partial V_s}{\partial \sigma_T^2} = \frac{\partial}{\partial \sigma_T^2}\left(\frac{\sigma_x^4}{\sigma_T^2} h(d)\right) = -\frac{\sigma_x^4}{(\sigma_T^2)^2} h(d) + \frac{\sigma_x^4}{\sigma_T^2} h'(d)\,\frac{\partial d}{\partial \sigma_T^2}.$$

By the definition of d and since $\tilde{u}_r^{s_r} - \tilde{u}_x^{s_x}$ does not depend on σ_T^2,

$$\frac{\partial d}{\partial \sigma_T^2} = \left(\tilde{u}_r^{s_r} - \tilde{u}_x^{s_x}\right)\left(-\frac{1}{2}\right)(\sigma_T^2)^{-3/2} = -\frac{\tilde{u}_r^{s_r} - \tilde{u}_x^{s_x}}{2(\sigma_T^2)^{3/2}} = -\frac{d}{2\sigma_T^2}.$$

where the last step uses again the definition of d. Substituting into the expression for $\partial V_s / \partial \sigma_T^2$ gives

$$\frac{\partial V_s}{\partial \sigma_T^2} = -\frac{\sigma_x^4}{(\sigma_T^2)^2} h(d) + \frac{\sigma_x^4}{\sigma_T^2} h'(d)\left(-\frac{d}{2\sigma_T^2}\right) = -\frac{\sigma_x^4}{(\sigma_T^2)^2}\left[h(d) + \tfrac{1}{2} d h'(d)\right].$$

Using $\partial \sigma_T^2 / \partial \sigma_r^2 = \partial \sigma_T^2 / \partial \sigma_o^2 = 1$ and the chain rule,

$$\frac{\partial V_s}{\partial \sigma_r^2} = \frac{\partial V_s}{\partial \sigma_T^2}\frac{\partial \sigma_T^2}{\partial \sigma_r^2} = \frac{\partial V_s}{\partial \sigma_T^2}, \qquad \frac{\partial V_s}{\partial \sigma_o^2} = \frac{\partial V_s}{\partial \sigma_T^2}\frac{\partial \sigma_T^2}{\partial \sigma_o^2} = \frac{\partial V_s}{\partial \sigma_T^2}.$$

Thus

$$\frac{\partial V_s}{\partial \sigma_r^2} = \frac{\partial V_s}{\partial \sigma_o^2} = -\frac{\sigma_x^4}{(\sigma_x^2 + \sigma_r^2 + \sigma_o^2)^2}\left[h(d) + \tfrac{1}{2}dh'(d)\right].$$

Let $q(d) := h(d) + \frac{1}{2}dh'(d)$ and note that the sign of $\frac{\partial V_s}{\partial \sigma_T^2}$ is the opposite of the sign of $q(d)$. Thus, all we are left with is to determine the sign of $q(d)$. We do so by analyzing its values at zero, its asymptotic behavior, and its roots.

Since h is even, $h'(0) = 0$. At $d = 0$, using $\Phi(0) = 1/2$ and $\phi(0) = 1/\sqrt{2\pi}$, we have

$$q(0) = h(0) = \frac{(1/\sqrt{2\pi})^2}{1/2 \cdot 1/2} = \frac{2}{\pi} > 0.$$

As $d \to \infty$, we use the standard Mill's ratio approximation $1 - \Phi(d) \approx \frac{\phi(d)}{d}$. This implies $h(d) \approx d\phi(d)$ and $h'(d) \approx \phi(d)(1 - d^2)$. Substituting these into $q(d)$ yields

$$q(d) \approx d\phi(d) + \frac{1}{2}d\left[\phi(d)(1 - d^2)\right] \approx -\frac{1}{2}d^3\phi(d).$$

Thus, for sufficiently large d, $q(d) < 0$. By the Intermediate Value Theorem, there exists at least one zero $z^\star$.

Next, we show that $z^\star$ is unique, that is, that q admits a unique root. Consider $S(d) := \frac{q(d)}{h(d)} = 1 + \frac{d}{2}(\ln h(d))'$. Since $h(d) > 0$, the roots of $q(d)$ are identical to the roots of $S(d)$. To prove that q admits a unique root it is therefore sufficient to prove that S has a unique root. Since we have already established that $q(0) > 0$ and $q(d) < 0$ for d large enough and the same must be true for S, then to prove that S has a unique root, it is sufficient to show that S is strictly decreasing on $(0, \infty)$.

Let $r(x) := \frac{\phi(x)}{1-\Phi(x)}$. Since $h(d) = \frac{\phi(d)^2}{\Phi(d)\Phi(-d)}$, then $\ln h(d) = \ln r(d) + \ln r(-d)$. Differentiating with respect to d and using the property $r'(x) = r(x)(r(x) - x)$, we obtain:

$$\frac{\partial \ln h(d)}{\partial d} = r(d) - r(-d) - 2d.$$

Let $g(d) := r(d) - r(-d) - 2d$. The derivative is $g'(d) = r'(d) + r'(-d) - 2$. Note that r is a well-known function and it is known that $0 < r'(x) < 1$ for all real x (see, e.g., Sampford, 1953). It follows that $g'(d) < 0$. Since $g(0) = 0$, it follows that $g(d)$ is negative and strictly decreasing for $d > 0$. Substituting back into $S(d)$, we obtain

$$S(d) = 1 + \frac{1}{2}d \cdot g(d).$$

Since $d > 0$ is increasing and $g(d) < 0$ and decreasing, the product $d \cdot g(d)$ is strictly decreasing.

Therefore, $S(d)$ is strictly decreasing. It follows that S has a unique root and so does q.

We have therefore established that $q(d) > 0$ for $|d| < z^\star$ and $q(d) < 0$ for $|d| > z^\star$, and that q admits a unique root. Since the sign of $\frac{\partial V_s}{\partial \sigma_T^2}$ is the opposite of the sign of $q(d)$, we have $\frac{\partial V_s}{\partial \sigma_r^2} = \frac{\partial V_s}{\partial \sigma_o^2} = \frac{\partial V_s}{\partial \sigma_T^2} < 0$ for d small enough and > 0 otherwise. Note also that the sign of $\frac{\partial V_s}{\partial \sigma_{p,r}^2}$ matches that of $\frac{\partial V_s}{\partial \sigma_r^2}$.

We are now left to derive the corresponding properties for $V(r)$, where $V(r) = \mathbb{E}_s[V_s(r)]$. We begin with the comparative statics w.r.t. σ_o^2. Since the distribution of s is independent of σ_o^2, we have $\frac{\partial V}{\partial \sigma_o^2} = \mathbb{E}_s\left[\frac{\partial V_s}{\partial \sigma_o^2}\right]$. We established that $\frac{\partial V_s}{\partial \sigma_o^2} < 0$ if and only if $|d| < z^*$. If the prior bias $|\tilde{u}_r - \tilde{u}_x|$ is small, the distribution of the random variable d is concentrated around 0. By continuity, for sufficiently small bias, the probability mass of d contained within $(-z^*, z^*)$ is sufficiently large such that $\mathbb{E}_s\left[\frac{\partial V_s}{\partial \sigma_o^2}\right] < 0$. Conversely, for a large bias, the mass of d shifts to the tails where the derivative is positive. This proves the statement for σ_o^2.

Finally, consider the comparative static with respect to the prior variance $\sigma_{p,r}^2$. The parameter $\sigma_{p,r}^2$ affects both σ_T^2 and the variance of the distribution of d. At zero bias, the distribution of d is centered at 0, so most mass lies in $|d| < z^*$, where the derivative is negative. Furthermore, increasing $\sigma_{p,r}^2$ increases the variance of d, spreading mass from the center (where V_s is maximized) to the tails (where V_s is lower), which reinforces the negative effect. Thus, by continuity, for sufficiently small bias, $V(r)$ is strictly decreasing in $\sigma_{p,r}^2$. Conversely, as bias grows to ∞, the probability mass of d concentrates entirely in the region where $|d| > z^*$ (indeed, $\Pr(|d| > z^*) \to 1$), where $\frac{\partial V_s}{\partial \sigma_T^2}$ is strictly positive. Thus, for sufficiently large bias, $V(r)$ is strictly increasing in $\sigma_{p,r}^2$. □

D.9 Proof of Observation 2

When $\sigma_{\epsilon,i}^2 \to \infty$, then $\frac{\lambda_i}{\bar{\sigma}_i} = \frac{\sigma_{p,i}}{\sigma_{\epsilon,i}\sqrt{\sigma_{\epsilon,i}^2 + \sigma_{p,i}^2}} \to 0$. Now notice that $\psi(0) = \frac{\bar{\sigma}_i}{\sqrt{2}}\sqrt{2/\pi} = \frac{\bar{\sigma}_i}{\sqrt{\pi}}$.

Moreover, as $\sigma_{\epsilon,i}^2 \to \infty$, we get $\bar{\sigma}_i \to \sigma_{p,i}^2$

Thus, the boost in that dimension is simply $\frac{\sigma_{p,i}^2}{\sqrt{\pi}}$. If $\sigma_{\epsilon,i}^2 \to \infty$ for all i, we obtain that the perceived utility is

$$E[u(x)|s_i, ord_i] = \sum_i \mu_i^x + \frac{\sigma_{p,i}^2}{\sqrt{\pi}}\left(|D_x| - |D_y|\right).$$

When the priors are the same on each dimensions of the two objects, and if the DM has the same prior variance across dimensions, he values x above y if and only if $|D_x| > |D_y|$. This will also be the case when the prior is uninformative, $\sigma_{p,i}^2 \to \infty$.

D.10 Proof of Proposition 7

For notational convenience we will replace u_x, u_y and u_z with x, y and z respectively.

We first show that

$$E(x-y|x>y) > E(x-y|x>y>z)$$

Note that, for any $\bar{y}$

$$\begin{aligned} E(x|x &> \bar{y}) = \mu_x + \sigma_x \frac{\phi\left(\frac{\mu_x-\bar{y}}{\sigma_y}\right)}{\Phi\left(\frac{\mu_x-\bar{y}}{\sigma_y}\right)} \\ &= \mu_x + \sigma_x \lambda\left(\frac{\mu_x-\bar{y}}{\sigma_y}\right) \end{aligned}$$

where $\phi(.)$ and $\Phi(.)$ are the PDF and CDF of a standard normal and $\lambda(.) = \frac{\phi(.)}{\Phi(.)}$ is the inverse mills ratio. Define

$$\begin{aligned} h(\bar{y}) &= E(x|x>\bar{y}) - \bar{y} \\ &= \mu_x + \sigma_x \lambda\left(\frac{\mu_x-\bar{y}}{\sigma_x}\right) - \bar{y} \end{aligned}$$

This implies

$$h'(\bar{y}) = -1 - \lambda'\left(\frac{\mu_x-\bar{y}}{\sigma_x}\right)$$

Next note that we can bound $\lambda'(t)$ below at -1. To do so, first note that

$$\begin{aligned} \lambda'(t) &= \frac{\Phi(t)\phi'(t) - \phi(t)^2}{\Phi(t)^2} \\ &= \frac{-t\phi(t)\Phi(t) - \phi(t)^2}{\Phi(t)^2} \\ & \quad -\lambda(t)(t+\lambda(t)) \end{aligned}$$

Second, note that the function

$$z_t(u) = \frac{e^{-\frac{1}{2}u^2}}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv}$$

is a PDF on $[-\infty, t]$, and has a variance equal to

$$
\begin{aligned}
& E(u^2) - (E(u))^2 \\
& \frac{1}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} \left[\int_{-\infty}^{t} u^2 e^{-\frac{1}{2}u^2} du \right] - \left(\frac{1}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} \right)^2 \left(\int_{-\infty}^{t} u e^{-\frac{1}{2}u^2} du \right)^2
\end{aligned}
$$

Setting $df = ue^{-\frac{1}{2}u^2}$ and $g = u$, noting that $\int ue^{-\frac{1}{2}u^2} du = -e^{-\frac{1}{2}u^2}$ and applying integration by parts gives

$$\int_{-\infty}^{t} u^2 e^{-\frac{1}{2}u^2} = \left[-ue^{-\frac{1}{2}u^2} \right]_{-\infty}^{t} + \int_{-\infty}^{t} e^{-\frac{1}{2}u^2} du \tag{32}$$

By L'Hopital's rule, the bracketed term evaluated at $-\infty$ is zero, so we have

$$\int_{-\infty}^{t} u^2 e^{-\frac{1}{2}u^2} = -te^{-\frac{1}{2}t^2} + \int_{-\infty}^{t} e^{-\frac{1}{2}u^2} du$$

and so

$$
\begin{aligned}
& \frac{1}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} \left[\int_{-\infty}^{t} u^2 e^{-\frac{1}{2}u^2} du \right] \\
= & \frac{-te^{-\frac{1}{2}t^2}}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} + \frac{\int_{-\infty}^{t} e^{-\frac{1}{2}u^2} du}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} \\
= & -t\lambda(t) + 1
\end{aligned}
$$

where we use the fact that we can rewrite the inverse mills ratio as $\lambda(t) = \frac{e^{-\frac{1}{2}t^2}}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv}$

Next, and again using the fact that $\int_{-\infty}^{t} ue^{-\frac{1}{2}u^2} du = -e^{-\frac{1}{2}u^2}$, we have

$$\left(\int_{-\infty}^{t} ue^{-\frac{1}{2}u^2} du \right)^2 = \left(\left[-e^{-\frac{1}{2}u^2} \right]_{-\infty}^{t} \right)^2$$

again, the limit as $u = -\infty$ is equal to zero, so that gives

$$\left(\int_{-\infty}^{t} ue^{-\frac{1}{2}u^2} du \right)^2 = \left(-e^{-\frac{1}{2}t^2} \right)^2$$

and

$$\left(\frac{1}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} \right)^2 \left(\int_{-\infty}^{t} ue^{-\frac{1}{2}u^2} du \right)^2 = \left(\frac{-e^{-\frac{1}{2}t^2}}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} \right)^2 = \lambda(t)^2$$

And so the variance of $z_t(u)$ is given by

$$1 - t\lambda(t) - \lambda(t)^2 > 0$$

Using the fact that $\lambda'(t) = -\lambda(t)(t + \lambda(t))$ gives

$$1 + \lambda'(t) > 0 \;\; \Rightarrow \lambda(t) > -1$$

Implying that $h'(\bar{y}) < 0$.

Next, we find an expression for $E(x - y|x > y)$. Define

$$\begin{aligned} p_A(\bar{y}) &= p(y = \bar{y}, x > y) = p(y = \bar{y})p(x > \bar{y}) \\ & \frac{1}{\sigma_y}\phi\left(\frac{\bar{y} - \mu_y}{\sigma_y}\right)\left(1 - \Phi\left(\frac{\bar{y} - \mu_x}{\sigma_x}\right)\right). \end{aligned}$$

Then

$$\begin{aligned} E(x - y|x &> y) = \int E(x - \bar{y}|x > y)p(y = \bar{y}|x > y)d(\bar{y}) \\ &= \int h(\bar{y})\frac{p(y = \bar{y}, x > y)}{x > y}d(\bar{y}) \\ &= \int h(\bar{y})\frac{p_A(\bar{y})}{\int p_A(y)d(y)}d(\bar{y}) \\ &= E(h(\bar{y})|x > y), \end{aligned}$$

noting that $p(y = \bar{y}|x > y) = \frac{p_A(\bar{y})}{\int p_A(y)d(y)}$.

Next we find an expression for $E(x - y|x > y > z)$. First note that

$$p(y = \bar{y}, x > y, y > z) = p_A(\bar{y})\Phi\left(\frac{\bar{y} - \mu_z}{\sigma_z}\right)$$

as z is independent of x and y. In turn,

$$p(x > y > z) = \int p_A(y)\Phi\left(\frac{y - \mu_z}{\sigma_z}\right)d(y).$$

We can then write

$$\begin{aligned} E(x-y|x>y>z) &= \int E(x-\bar{y}|x>\bar{y}>z)p(y=\bar{y}|x>y>z)d(\bar{y}) \\ &= \int E(x-\bar{y}|x>\bar{y}>z)\frac{p(y=\bar{y},x>y,y>z)}{p(x>y>z)}d(\bar{y}) \\ &= \int E(x-\bar{y}|x>\bar{y}>z)\frac{p_A(\bar{y})\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)}{\int p_A(y')\Phi\left(\frac{y'-\mu_z}{\sigma_z}\right)d(y')}d(\bar{y}). \end{aligned}$$

Dividing through top and bottom by $\int p_A(y)d(y)$ gives

$$E(x-y|y>x>z) = \frac{\int E(x-\bar{y}|x>\bar{y}>z)\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)\frac{p_A(\bar{y})}{\int p_A(y)d(y)}d(\bar{y})}{\int \frac{p_A(y')}{\int p_A(y)d(y)}\Phi\left(\frac{y'-\mu_z}{\sigma_z}\right)d(y')}$$

Noting that for a fixed $\bar{y}$ $E(x-\bar{y}|x>\bar{y}>z) = E(x-\bar{y}|x>\bar{y}) = h(\bar{y})$ we have

$$\begin{aligned} &= \frac{\int h(\bar{y})\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)p(y=\bar{y}|x>y)d(\bar{y})}{\int \Phi\left(\frac{y'-\mu_z}{\sigma_z}\right)p(y=y'|x>y)d(y')} \\ &= \frac{E\left(h(\bar{y})\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)|x>y\right)}{E\left(\Phi\left(\frac{y'-\mu_z}{\sigma_z}\right)|x>y\right)}. \end{aligned}$$

Recall that we have shown that h is strictly decreasing; $\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)$ is also a strictly increasing function of $\bar{x}$. Then, Chebyshev's covariance inequality (see e.g. Agahi (2015)) guarantees that[48]

$$cov\left(h(\bar{y}),\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)|x>y\right) < 0,$$

with the strict inequality guaranteed by the fact that that h and ϕ are strictly monotonic and $\bar{y}$ is not almost surely constant. This means

$$cov\left(h(\bar{y}),\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)|x>y\right) = E\left(h(\bar{y})\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)|x>y\right) - E(h(\bar{y})|x>y)E\left(\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)|x>y\right) < 0$$

[48]Chebyshev's inequality states that for two strictly increasing functions $f(x)$ and $g(x)$ their covariance must be weakly positive. However, as $cov(-f(x),g(x)) = -cov(f(x),g(x))$ we obtain the required result.

thus (since $E\left(\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)|y > x\right)$)

$$\frac{E\left(h(\bar{y})\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)|x > y\right)}{E\left(\Phi\left(\frac{\bar{y}-\mu_z}{\sigma_z}\right)|x > y\right)} < E(h(\bar{y})|x > y)$$

hence $E(x - y|x > y > z) < E(x - y|x > y)$, completing the proof.

A similar argument shows that $E(x - y|x > y) > E(x - y|z > x > y)$. Define

$$\begin{aligned} g(\bar{x}) &= E(\bar{x} - y|x > y) \\ &= \bar{x} - \mu_y + \sigma_y \frac{\phi\left(-\frac{\mu_y - \bar{x}}{\sigma y}\right)}{\Phi\left(-\frac{\mu_y - \bar{x}}{\sigma_y}\right)} \\ &= \bar{x} - \mu_y + \sigma_y \lambda\left(-\frac{\mu_y - \bar{x}}{\sigma_y}\right) \end{aligned}$$

Taking derivatives with respect to $\bar{x}$ gives

$$g'(\bar{x}) = 1 + \lambda'\left(-\frac{\mu_y - \bar{x}}{\sigma_y}\right)$$

As argued above, we can bound $\lambda'(t)$ below by -1, meaning that $g'(\bar{x})$ is strictly increasing. Again note that we can write

$$\begin{aligned} p_B(\bar{x}) &= p(x = \bar{x}, x > y) = p(x = \bar{x})p(\bar{x} > y) \\ & \frac{1}{\sigma_x}\phi\left(\frac{\bar{x} - \mu_x}{\sigma x}\right)\Phi\left(\frac{\bar{x} - \mu_y}{\sigma_y}\right). \end{aligned}$$

and

$$\begin{aligned} E(x - y|x &> y) = \int E(\bar{x} - y|x > y)p(x = \bar{x}|x > y)d(\bar{x}) \\ &= \int g(\bar{x})\frac{p_B(\bar{x})}{\int p_B(x)d(x)}d(\bar{x}) = \\ &= E(g(\bar{x})|x > y), \end{aligned}$$

Now note that

$$p(x = \bar{x}, z > x, x > y) = p_B(\bar{x})\left(1 - \Phi\left(\frac{\bar{x} - \mu_z}{\sigma_z}\right)\right).$$

and, following the procedure from the first part of the proof we get

$$
\begin{aligned}
E(x-y|z &> x>y) = \int E(\bar{x}-y|z>\bar{x}>y)p(x=\bar{x},z>x,x>y)d(\bar{x}) \\
&= \frac{E\left(g(\bar{x})\left(1-\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)\right)|x>y\right)}{E\left(1-\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)|x>y\right)}.
\end{aligned}
$$

By the same argument as before we have that, as $g(\bar{x})$ is increasing and $\left(1-\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)\right)$ is decreasing we have

$$
cov\left(g(\bar{x}),\left(1-\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)\right)|x>y\right) = E\left(g(\bar{x})\left(1-\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)\right)|x>y\right) - E(g(\bar{x})|x>y)E\left(\left(1-\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)\right)|x
$$

and so

$$
E(x-y|x>y) = E(g(\bar{x})|x>y) > \frac{E\left(g(\bar{x})\left(1-\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)\right)|x>y\right)}{E\left(1-\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)|x>y\right)} = E(x-y|z>x>y)
$$

Finally we need to show that $E(x-y|x>z>y) > E(x-y|x>y)$.Let

$$
h(x,y) := x-y \quad \text{and} \quad q(x,y) := p(z \in (y,x)) = \Phi\left(\frac{x-\mu_z}{\sigma_z}\right) - \Phi\left(\frac{y-\mu_z}{\sigma_z}\right).
$$

Note that, conditional on $x > y$, $h(x,y)$ and $q(x,y)$ are both greater than zero, and strictly increasing in their first and strictly decreasing in their second argument. This in turn means that both are strictly increasing functions of $(x,-y)$. We wish to show that this implies that that the covariance of the two functions must be positive conditional on $x > y$. To do so, note that using the Law of Total Covariance, we can write

$$
cov(h(x,y),q(x,y)|x>y) = E_{\bar{y}}(cov(h(x,\bar{y}),q(x,\bar{y})|x>y,\bar{y})) + cov_{\bar{y}}\left(E(h(x,\bar{y})|x>y,\bar{y}),E(q(x,\bar{y})|x>y,\bar{y})\right)
$$

Notice that for a fixed $\bar{y}$, $h(x,\bar{y})$ and $q(x,\bar{y})$ are both strictly increasing functions of a non degenerate random variable x, and so $cov(h(x,\bar{y}),q(x,\bar{y})|x>y,\bar{y})$ is strictly greater than zero (again using Chebyshev's covariance inequality). Thus the first term is strictly positive. Let $A(\bar{y}) = E(h(x,\bar{y})|x>y,\bar{y})$ and $B(\bar{y}) = E(q(x,\bar{y})|x>y,\bar{y})$ and note that, as x and y are independent, and both h and q are strictly decreasing in $\bar{y}$, the covariance of A and B is weakly positive. Thus we can conclude that

$$cov(h(x,y),q(x,y)|x>y)>0$$

Now note that

$$p(x=\bar{x},y=\bar{y}|x>z>y)=\frac{p(x=\bar{x},y=\bar{y}|x>y)q(\bar{x},\bar{y})|x>y)}{p(x>z>y|x>y)}.$$

So

$$\begin{aligned}
E(x-y|x>z>y) &= E(h(x,y)|x>z>y)=\int h(\bar{x},\bar{y})p(x=\bar{x},y=\bar{y}|x>z>y)d(\bar{x},\bar{y}) \\
&= \int h(\bar{x},\bar{y})\frac{p(x=\bar{x},y=\bar{y}|x>y)q(\bar{x},\bar{y}|x>y)}{p(x>z>y|x>y)}d(\bar{x},\bar{y}) \\
&= \int h(\bar{x},\bar{y})\frac{p(x=\bar{x},y=\bar{y}|x>y)q(\bar{x},\bar{y}|x>y)}{\int p(x=x',y=y'|x>y)q(x',y'|x>y)d(x',y')}d(\bar{x},\bar{y}) \\
&= \frac{E(h(x,y)q(x,y)|x>y)}{E(q(x,y)|x>y)}.
\end{aligned}$$

Note that

$$\begin{aligned}
cov(h(x,y),q(x,y)|x>y) &> 0 \Rightarrow \\
E(h(x,y)q(x,y)|x>y)-E(h(x,y)|x>y)E(q(x,y)|x>y) &> 0 \Rightarrow \\
\frac{E(h(x,y)q(x,y)|x>y)}{E(q(x,y)|x>y)} &> E(h(x,y)|x>y)
\end{aligned}$$

since $E(q(x,y)|x>y)>0$. But then

$$E(x-y|x>z>y)=\frac{E(h(x,y)q(x,y)|x>y)}{E(q(x,y)|x>y)}>E(h(x,y)|x>y)=E(x-y|x>y),$$

completing the proof.

D.11 Proof of Proposition 8

The proposition is a special case of Proposition 16 and from which it immediately follows. □

D.12 Proof of Proposition 9

Let $\mathcal{I}$ denote the collection of all (cardinal and ordinal) signals the DM receives before choosing between x and y, and write

$$\hat{U}_a := E[U(a) \mid \mathcal{I}], \qquad a \in \{x, y\}.$$

Step 1: Decomposition of the objective. Given $\mathcal{I}$, the DM chooses the option with the larger posterior mean, so his ex–ante expected utility is

$$U_{\max}(r) := E\big[\max\{\hat{U}_x, \hat{U}_y\}\big].$$

Using $\max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$, we obtain

$$U_{\max}(r) = \frac{1}{2}E[\hat{U}_x + \hat{U}_y] + \frac{1}{2}E\big[|\hat{U}_x - \hat{U}_y|\big].$$

Since $E[\hat{U}_a] = E[U(a)]$ for $a \in \{x, y\}$ and since these quantities depend only on the priors, not on r, then maximizing $U_{\max}(r)$ is equivalent to maximizing

$$E\big[|\hat{U}_x - \hat{U}_y|\big].$$

Step 2: Separability and mean–preserving spreads. Since the utility is additive, $U(a) = \sum_{i=1}^{n} u_{a,i}$ and, because priors and signals factor by dimension,

$$\hat{U}_a = \sum_{i=1}^{n} \hat{u}_{a,i}, \qquad \hat{u}_{a,i} := E[u_{a,i} \mid \mathcal{I}_i],$$

where $\mathcal{I}_i$ collects the signals in dimension i. Define the posterior difference in dimension i by

$$D_i(r_i) := \hat{u}_{x,i} - \hat{u}_{y,i},$$

and the total posterior difference by $D(r) := \hat{U}_x - \hat{U}_y = \sum_{i=1}^{n} D_i(r_i)$. For each i, r_i affects only $D_i(r_i)$, and the vectors $\big(D_i(r_i)\big)_{i=1}^{n}$ are independent across dimensions.

Recall that $|\cdot|$ is convex. We will use the following simple fact. Let X' be a mean–preserving spread of X and let Y be independent of (X, X'). Then, for every convex φ, $E[\varphi(X' + Y)] \geq E[\varphi(X + Y)]$. (This follows from conditioning on $Y = y$ and using the fact that $x \mapsto \varphi(x + y)$ is convex for each y.)

Applied with $\varphi(z) = |z|$, this implies that if we replace $D_i(r_i)$ by a mean–preserving spread $D_i'(r_i)$, holding all other dimensions fixed, then $E[|D(r)|]$ strictly increases. Therefore, to maximize $U_{\max}(r)$ it suffices to choose each r_i so that $D_i(r_i)$ is as dispersed (in the sense of mean–preserving spreads) as possible. We henceforth focus on a single dimension i (and drop the index from all subscripts).

Step 3: The reference point and mean–preserving spreads. Because ordinal signals with respect to r are noiseless, the realization of ordinal signals partitions the state space into three events:

$$\begin{aligned} A^- &:= \{u_x > u_y > u_r\} \cup \{u_y > u_x > u_r\}, \\ A^+ &:= \{u_r > u_x > u_y\} \cup \{u_r > u_y > u_x\}, \\ A^0 &:= \{u_y < u_r < u_x\} \cup \{u_x < u_r < u_y\}. \end{aligned}$$

Conditional on all other (cardinal and pairwise-ordinal) signals in this dimension, (u_x, u_y, u_r) are independent Normal random variables. Applying Proposition 7 to (u_x, u_y, u_r), first on the event $\{u_x > u_y\}$ and then on $\{u_y > u_x\}$ (swapping the roles of x and y), yields the strict inequalities

$$E\big[|u_x - u_y| \mid A^\pm\big] \;<\; E\big[|u_x - u_y| \mid u_x \neq u_y\big] \;<\; E\big[|u_x - u_y| \mid A^0\big].$$

Since $D(r)$ is the conditional expectation of $u_x - u_y$ given all signals, the same ordering holds for $|D(r)|$:

$$E\big[|D(r)| \mid A^\pm\big] \;<\; E\big[|D(r)| \mid u_x \neq u_y\big] \;<\; E\big[|D(r)| \mid A^0\big].$$

Moreover, the unconditional mean $E[D(r)] = E[u_x - u_y]$ is independent of r, because the reference point affects only the information structure, not the prior.

Thus, as we vary u_r, moving probability mass from $A^\pm$ (where $|D(r)|$ is relatively small) to A^0 (where $|D(r)|$ is relatively large), while keeping $E[D(r)]$ fixed, yields a (strict) mean–preserving spread of $D(r)$. By Step 2, this strictly increases $E|D(r)|$ and hence $U_{\max}(r)$. Consequently, in each dimension, we want to choose u_r so as to maximize

$$P_{\text{split}}(u_r) := P(u_y < u_r < u_x) + P(u_x < u_r < u_y),$$

the probability that u_r lies between u_x and u_y ("splits" the two options).

Step 4: Maximizing the splitting probability. Fix a dimension and write $\tilde{u}_x, \tilde{u}_y$ for the prior means and σ^2 for the common prior variance. Let $X \sim N(\tilde{u}_x, \sigma^2)$ and $Y \sim N(\tilde{u}_y, \sigma^2)$ be independent.

Then

$$P_{\text{split}}(u_r) = P(X > u_r)P(Y < u_r) + P(X < u_r)P(Y > u_r),$$

since the events for X and Y are independent. Let Φ and ϕ denote the standard Normal cdf and pdf, and set

$$z_x := \frac{u_r - \tilde{u}_x}{\sigma}, \qquad z_y := \frac{u_r - \tilde{u}_y}{\sigma}.$$

Then

$$\begin{aligned} P_{\text{split}}(u_r) &= [1 - \Phi(z_x)]\Phi(z_y) + \Phi(z_x)[1 - \Phi(z_y)] \\ &= \Phi(z_x) + \Phi(z_y) - 2\,\Phi(z_x)\Phi(z_y). \end{aligned}$$

Differentiating with respect to u_r gives

$$P'_{\text{split}}(u_r) = \frac{1}{\sigma}\phi(z_y)\big[1 - 2\Phi(z_x)\big] + \frac{1}{\sigma}\phi(z_x)\big[1 - 2\Phi(z_y)\big].$$

Write $\Delta := \tilde{u}_x - \tilde{u}_y \neq 0$ and assume, without loss of generality, that $\tilde{u}_x > \tilde{u}_y$. Consider the midpoint

$$u_r^* := \frac{\tilde{u}_x + \tilde{u}_y}{2}.$$

At this point, $z_x = -z_y$ and therefore $\phi(z_x) = \phi(z_y)$ and $\Phi(z_x) + \Phi(z_y) = 1$. Substituting into the derivative,

$$\begin{aligned} P'_{\text{split}}(u_r^*) &= \frac{\phi(z_x)}{\sigma}\Big[(1 - 2\Phi(z_x)) + (1 - 2\Phi(z_y))\Big] \\ &= \frac{\phi(z_x)}{\sigma}\big[2 - 2(\Phi(z_x) + \Phi(z_y))\big] = 0. \end{aligned}$$

A second derivative calculation (omitted for brevity) shows that $P''_{\text{split}}(u_r^*) < 0$, whereas $P_{\text{split}}(u_r) \to 0$ as $u_r \to \pm\infty$. Hence u_r^* is the unique maximizer of P_{split} in this dimension:

$$u_r^* = \frac{1}{2}\big(\tilde{u}_x + \tilde{u}_y\big).$$

Since the dimensions are independent and utility is additive, the argument applies dimension by dimension. Thus $U_{\max}(r)$ is maximized by choosing, for every i,

$$u_{r,i} = \frac{1}{2}\big(\tilde{u}_{x,i} + \tilde{u}_{y,i}\big).$$

Step 5: Sufficiency of the Splitting Probability via Symmetry. It remains to show that maximizing

$P_{\text{split}}(u_r)$ maximizes the total objective $U_{\max}(r)$. Let $M(u_r) = \mathbb{E}[|\hat{U}_x - \hat{U}_y|]$. Note that the problem is symmetric around the midpoint $m = (\tilde{u}_x + \tilde{u}_y)/2$. Specifically, consider a reference point $u_r = m + \delta$. Due to the symmetry of the Normal priors and the boost function $\psi(\cdot)$, the distribution of the posterior difference $\hat{U}_x - \hat{U}_y$ under $u_r = m + \delta$ is identical to the distribution of $\hat{U}_y - \hat{U}_x$ under $u_r = m - \delta$. Since the objective function depends on the *absolute* difference $|\hat{U}_x - \hat{U}_y|$, it is invariant to this reflection: $M(m + \delta) = M(m - \delta)$.

This symmetry implies that $u_r = m$ is a critical point of the objective function. To confirm it is a maximum, recall that the total expectation is the sum of the expected differences conditional on splitting and not splitting. We established in Step 4 that the probability of the splitting event (where the ordinal signals force the posteriors apart, $\hat{U}_x > \hat{U}_y$) is strictly concave and uniquely maximized at $u_r = m$. Conversely, the probabilities of the non-splitting events (where boosts are in the same direction and partially cancel out, reducing $|\hat{U}_x - \hat{U}_y|$) are minimized at $u_r = m$. Since the "split" event generates the largest ordinal separation, the maximization of the probability weight on this event, combined with the symmetry of the conditional magnitudes around m, ensures that $u_r = m$ globally maximizes the ex-ante expected utility. □

E Proofs of the Results in the Appendix

E.1 Proof of Proposition 10

Since X and Y are jointly normally distributed, and E is independent of both, then X and S are jointly normally distributed with

$$\begin{aligned} \sigma_s^2 &= \sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY} + \sigma_E^2 \\ \mathrm{Cov}(X, S) &= \sigma_{XS} = \sigma_X^2 - \sigma_{XY} \end{aligned}$$

A standard property of the multivariate normal distribution is that

$$\mathbb{E}[X \mid S = s] = \mu_X + \frac{\mathrm{Cov}(X, S)}{\sigma_S^2}(s - \mu_S).$$

Taking conditional expectations again gives

$$
\begin{aligned}
\mathbb{E}[X \mid S > 0] &= \mathbb{E}[\,\mathbb{E}[X \mid S] \mid S > 0] \\
&= \mathbb{E}\left[\mu_X + \frac{\mathrm{Cov}(X,S)}{\sigma_S^2}(S - \mu_S) \,\middle|\, S > 0\right] \\
&= \mu_X + \frac{\mathrm{Cov}(X,S)}{\sigma_S^2}\left(\mathbb{E}[S \mid S > 0] - \mu_S\right).
\end{aligned}
$$

By the standard formula of the expectation of a trunated normal

$$
\mathbb{E}[S \mid S > 0] = \mu_S + \sigma_S \frac{\phi(t)}{\Phi(t)}
$$

where $t = \frac{\mu_s}{\sigma_s}$. Thus we have

$$
\begin{aligned}
\mathbb{E}[X \quad | \quad S > 0] &= \mu_X + \frac{\mathrm{Cov}(X,S)}{\sigma_S}\frac{\phi(t)}{\Phi(t)} \\
&= \quad \mu_X + \frac{\sigma_X^2 - \sigma_{XY}}{\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY} + \sigma_E^2}} \frac{\phi\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2+\sigma_Y^2-2\sigma_{XY}+\sigma_E^2}}\right)}{\Phi\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2+\sigma_Y^2-2\sigma_{XY}+\sigma_E^2}}\right)}
\end{aligned}
$$

A symmetric argument provides the formula for $\mathbb{E}[X \mid S < 0]$. □

E.2 Proof of Observation 3

Fix the common priors $u_x \sim \mathcal{N}(\tilde{u}_x, \sigma_x^2)$ and $u_r \sim \mathcal{N}(\tilde{u}_r, \sigma_r^2)$, $\sigma_v^2 = \sigma_{v'}^2$, and recall that v and v' are independent of all other random variables. Since o and o' have the same conditional distribution given (u_x, u_r), it suffices to show that the cardinal signals alone induce the same posterior over (u_x, u_r).

Step 1: Reduce the Alternative Setting by an invertible transformation. In the Alternative Setting, define the residual error $t := e_d - e_x - e_r$. Under the stated covariance restrictions, $t \sim \mathcal{N}(0, \sigma_t^2)$ with $\sigma_t^2 = \sigma_{e_d}^2 - \sigma_{e_x}^2 - \sigma_{e_r}^2 > 0$, and t is independent of (e_x, e_r). Define the transformed signal

$$
z \;:=\; s_x' - s_d \;=\; (u_x + e_x) - (u_x - u_r + e_d) \;=\; u_r - e_r - t.
$$

The mapping $(s_x', s_r', s_d) \mapsto (s_x', s_r', z)$ is invertible (since $s_d = s_x' - z$), so it preserves the agent's information set. Conditional on the states (u_x, u_r), the signals are

$$s'_x = u_x + e_x, \qquad (s'_r, z) = (u_r + e_r,\ u_r - e_r - t).$$

Since e_x is independent of (e_r, t), the belief update regarding u_x (via s'_x) is independent of the update regarding u_r (via s'_r and z).

Step 2: Extract an equivalent single Gaussian signal about u_r. The agent holds two signals about u_r: s'_r and z. Consider the further invertible transformation to the pair $(s'_r, \bar{s}_r)$, where

$$\bar{s}_r \ := \ \frac{s'_r + z}{2} \ = \ \frac{(u_r + e_r) + (u_r - e_r - t)}{2} \ = \ u_r - \frac{t}{2}.$$

Conditional on u_r, the signal s'_r has error e_r and the signal $\bar{s}_r$ has error $-t/2$. Because e_r and t are independent, the agent effectively holds two independent Gaussian signals about u_r with independent errors with precisions $1/\sigma^2_{e_r}$ and $1/(\sigma^2_t/4) = 4/\sigma^2_t$, respectively. By sufficiency, these are equivalent to a single Gaussian signal $s^*_r = u_r + \varepsilon^*_r$ with precision

$$\frac{1}{\sigma^2_{\varepsilon_r}} = \frac{1}{\sigma^2_{e_r}} + \frac{4}{\sigma^2_t} = \frac{1}{\sigma^2_{e_r}} + \frac{4}{\sigma^2_{e_d} - \sigma^2_{e_x} - \sigma^2_{e_r}}.$$

Step 3: Conclude Alternative $\longrightarrow$ Original. Combining Steps 1 and 2, the Alternative Setting yields the same information about (u_x, u_r) as observing

$$s'_x = u_x + e_x \quad \text{and} \quad s^*_r = u_r + \varepsilon^*_r,$$

with independent errors $e_x \sim \mathcal{N}(0, \sigma^2_{e_x})$ and $\varepsilon^*_r \sim \mathcal{N}(0, \sigma^2_{\varepsilon_r})$. Together with the ordinal signal, this induces the same posterior as the Original Setting with parameters defined in item 1 of the Observation.

Step 4: Conclude Original $\longrightarrow$ Alternative. Conversely, fix an Original Setting $(\sigma^2_{\varepsilon_x}, \sigma^2_{\varepsilon_r}, \sigma^2_v)$. Set $\sigma^2_{e_x} = \sigma^2_{\varepsilon_x}$ and $\sigma^2_{v'} = \sigma^2_v$. Choose any $\sigma^2_{e_r} > \sigma^2_{\varepsilon_r}$ and define

$$\sigma^2_t := 4\left(\frac{1}{\sigma^2_{\varepsilon_r}} - \frac{1}{\sigma^2_{e_r}}\right)^{-1} \ > \ 0, \qquad \sigma^2_{e_d} := \sigma^2_{e_x} + \sigma^2_{e_r} + \sigma^2_t.$$

Let (e_x, e_r, t) be independent Gaussians with these variances and set $e_d = e_x + e_r + t$. Then (e_x, e_d, e_r) satisfies the covariance structure of the Alternative Setting. By the equivalence established in Steps 1–3, this Alternative Setting yields the same posterior beliefs as the Original Setting. $\square$

E.3 Proof of Proposition 11

Mark: Check this proof

Given that the order in which signals are received does not effect beliefs, we will assume that decision maker first receives the cardinal signal about x then the cardinal comparative signal d, then the ordinal signal o.

Following the receipt of the signal about x they believe it to be distributed according to $N\left(\hat{\mu}_x^s, \sigma_x^2\right)$, as defined in Section 3.3. Note that x, r and d are jointly normally distributed with the covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_x^2 & 0 & \sigma_x^2 \\ 0 & \sigma_r^2 & -\sigma_r^2 \\ \sigma_x^2 & -\sigma_r^2 & \sigma_x^2 + \sigma_r^2 + \sigma_\varepsilon^2 \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

where

$$\Sigma_{11} = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$$

Conditional on d, x and r remain jointly normally distributed with

$$\begin{matrix} \hat{u}_x^{s,d} \\ \hat{u}_r^{s,d} \end{matrix} = \begin{matrix} \hat{u}_x^{s} \\ \hat{u}_r^{s} \end{matrix} + \Sigma_{12}\Sigma_{22}^{-1}\left(d - \left(\hat{\mu}_x^s - \mu_r\right)\right)$$

and

$$\bar{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}$$

where $\bar{\Sigma}$ is the covariance matrix of x and r conditional on d. Thus,

$$\begin{aligned} \hat{u}_x^{s,d} &= \left(1 - \frac{\sigma_x^2}{\sigma_x^2 + \sigma_r^2 + \sigma_\varepsilon^2}\right)\hat{\mu}_x^s + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_r^2 + \sigma_\varepsilon^2}(d + \mu_r) \\ \hat{u}_r^{s,d} &= \left(1 - \frac{\sigma_r^2}{\sigma_x^2 + \sigma_r^2 + \sigma_\varepsilon^2}\right)\mu_r - \frac{\sigma_x^2}{\sigma_x^2 + \sigma_r^2 + \sigma_\varepsilon^2}(d - \hat{\mu}_x^s) \end{aligned}$$

and

$$\begin{aligned} \bar{\Sigma} &= \begin{pmatrix} \bar{\sigma}_x^2 & \bar{\sigma}_{xr}^2 \\ \bar{\sigma}_{xr}^2 & \bar{\sigma}_r^2 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_x^2\left(1 - \frac{\bar{\sigma}_x^2}{\sigma_x^2+\sigma_r^2+\sigma_\varepsilon^2}\right) & \frac{-\sigma_x^2\sigma_r^2}{\sigma_x^2+\sigma_r^2+\sigma_\varepsilon^2} \\ \frac{-\sigma_x^2\sigma_r^2}{\sigma_x^2+\sigma_r^2+\sigma_\varepsilon^2} & \sigma_r^2\left(1 - \frac{\bar{\sigma}_r^2}{\sigma_x^2+\sigma_r^2+\sigma_\varepsilon^2}\right) \end{pmatrix} \end{aligned}$$

To complete the proof, all that is needed is to apply the result of Proposition 10. □

E.4 Proof of Proposition 12

Let $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$ and $e \sim N(0, \sigma_e^2)$ be distributed normally and independently, and let $S = X - Y + e$. We wish to derive the variance of X conditional on $S > 0$.

Note that X and S are jointly normal with the covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \sigma_X^2 \\ \sigma_X^2 & \sigma_X^2 + \sigma_Y^2 + \sigma_e^2 \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & \sigma_{XS} \\ \sigma_{XS} & \sigma_S^2 \end{pmatrix}$$

By the standard formula for the conditional mean variance of a bivariate normal we have that

$$\begin{aligned} E(X|S) &= \mu_X + \frac{\sigma_{XS}}{\sigma_S^2}(s - (\mu_X - \mu_Y)) \\ var(X|S) &= \sigma_X^2 - \frac{\sigma_{XS}^2}{\sigma_S^2} \end{aligned}$$

let $t = \frac{\mu_X - \mu_Y}{\sigma_S}$ and $\psi(x.+) = \frac{\phi(x)}{\Phi(x)}$. By the standard formula for the variance of a truncated normal distribution we have

$$var(S|S > 0) = \sigma_S^2(1 - t\psi(t) - \psi(t)^2)$$

Then using the law of total variance to get

$$var(X|S > 0) = E(var(X|S)|S > 0) + var(E(X|S)|S > 0)$$

Given that $var(X|S)$ is not a function of S, the first term equals $\sigma_X^2 - \frac{\sigma_{XS}^2}{\sigma_S^2}$. The second term is given by

$$\begin{aligned} var(E(X|S)|S &> 0) = var(\mu_X + \frac{\sigma_{XS}}{\sigma_S^2}(s-(\mu_X-\mu_Y))|s>0) \\ &= \frac{\sigma_{XS}^2}{\sigma_S^4} var\left(s|s>0\right) \\ &= \frac{\sigma_{XS}^2}{\sigma_S^4}\sigma_S^2(1-t\psi(t,+)-\psi(t,+)^2) \\ &= \frac{\sigma_X^4}{\sigma_S^2}(1-t\psi(t)-\psi(t)^2) \end{aligned}$$

Summing the two terms gives

$$\begin{aligned} var(X|S &> 0) = \sigma_X^2 - \frac{\sigma_X^4}{\sigma_S^2}(t\psi(t,+)+\psi(t,+)^2) \\ &= \sigma_X^2 - \frac{\sigma_X^4}{\sigma_X^2+\sigma_Y^2+\sigma_e^2}\left(t\psi(t,+)+\psi(t,+)^2\right) \end{aligned}$$

An equivalent argument gives that, if we learn that $S<0$ we have

$$var(X|S<0) = \sigma_X^2 - \frac{\sigma_X^4}{\sigma_S^2}(t\psi(t,-)+\psi(t,-)^2)$$

where $\psi(x.-) = -\frac{\phi(-x)}{\Phi(-x)}$

E.5 Proof of Proposition 13

We wish to calculate

$$\begin{aligned} E_{s,o}[\tilde{u}_x^{s,o}] &= \\ & E_{s,o}\left[\bar{u}_x^s + \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_0^2}}\psi(z_s,o)\right] \end{aligned}$$

where

$$z_s = \frac{\tilde{u}_x^s - \tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_0^2}} = \frac{\lambda s + (1-\lambda)\tilde{u}_x - \tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_0^2}}$$

Note that o does not depend on s, but only whether v_x is greater or less that $u_x - u_r$. We can therefore rewrite

$$
\begin{aligned}
& E_{s,o}\left[\bar{u}_x^s + \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_0^2}}\psi(z_s,o)\right] \\
= \;& \lambda u_x + (1-\lambda)\tilde{u}_x + \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_0^2}} E_s\left[\Phi\left(\frac{u_x-u_r}{\sigma_o}\right)\psi(z_s,+) + \left(1-\Phi\left(\frac{u_x-u_r}{\sigma_o}\right)\right)\psi(z_s,-)\right]
\end{aligned}
$$

Now note that $\psi(z_s,+) = \frac{\phi(z_s)}{\Phi(z_s)}$ and $\psi(z_s,-) = -\frac{\phi(-z_s)}{\Phi(-z_s)} = -\frac{\phi(z_s)}{1-\Phi(z_s)}$. For notational convenience let $\Phi\left(\frac{u_x-u_r}{\sigma_o}\right) - a$, then the term inside the remaining expectation can be written as

$$
\begin{aligned}
& a\frac{\phi(z_s)}{\Phi(z_s)} - (1-a)\,\frac{\phi(z_s)}{1-\Phi(z_s)} \\
= \;& \frac{a\phi(z_s) - a\phi(z_s)\Phi(z_s) - \phi(z_s)\Phi(z_s) + a\phi(z_s)\Phi(z_s)}{\Phi(z_s)\,(1-\Phi(z_s))} \\
= \;& \frac{\phi(z_s)\,(a-\Phi(z_s))}{\Phi(z_s)\,(1-\Phi(z_s))}
\end{aligned}
$$

Substituting back in gives

$$
E_{s,o}[\tilde{u}_x^{s,o}] = \lambda u_x + (1-\lambda)\tilde{u}_x + \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_0^2}} E_S\left[\frac{\phi(z_s)}{\Phi(z_s)\,(1-\Phi(z_s))}\left(\Phi\left(\frac{u_x-u_r}{\sigma_o}\right) - \Phi(z_s)\right)\right]
$$

E.6 Proof of Proposition 14

Fix s and write $\Delta_s := \tilde{u}_x^s - \tilde{u}_r$, $\tau := \sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}$, and $\rho(z) := \frac{\phi(z)}{\Phi(z)}$. Conditional on s, $u_x - u_r - \nu \mid s$ is normal with mean Δ_s and variance τ^2. By Proposition 10,

$$
\mathbb{E}[u_x \mid s, o=+] = \tilde{u}_x^s + \frac{\mathrm{Cov}(u_x, u_x-u_r-\nu \mid s)}{\sqrt{\mathbb{V}(u_x-u_r-\nu \mid s)}}\,\rho\left(\frac{\Delta_s}{\tau}\right).
$$

Since $\mathrm{Cov}(u_x, u_x-u_r-\nu \mid s) = \sigma_x^2$ and $\mathbb{V}(u_x-u_r-\nu \mid s) = \tau^2$,

$$
B(s,+) := \mathbb{E}[u_x \mid s, o=+] - \tilde{u}_x^s = \frac{\sigma_x^2}{\tau}\,\rho\left(\frac{\Delta_s}{\tau}\right) = \frac{\sigma_x^2}{\tau}\,\rho(z_s),
$$

which is strictly positive because $\sigma_x^2 > 0$, $\tau > 0$, and $\rho(\cdot) > 0$.

Part (a) Write $a := \sigma_x^2$ and $K := \sigma_r^2 + \sigma_o^2$, so $\tau = \sqrt{a+K}$ and $z_s = \Delta_s/\tau$. Using $\rho'(z) =$

$-\rho(z)\big(z+\rho(z)\big)$, differentiate $B(s,+)=\frac{a}{\tau}\rho(z_s)$ with respect to a (holding Δ_s fixed):

$$\frac{\partial B(s,+)}{\partial a} = \frac{1}{2\tau^3}\Big[(a+2K)\rho(z_s) - a z_s \rho'(z_s)\Big].$$

Substituting $\rho'(z_s) = -\rho(z_s)(z_s+\rho(z_s))$ yields

$$\frac{\partial B(s,+)}{\partial a} = \frac{\rho(z_s)}{2\tau^3}\Big[a\big(1+z_s^2+z_s\rho(z_s)\big)+2K\Big] \ > \ 0,$$

because $\rho(z_s)>0$, $K\geq 0$, and $1+z^2+z\rho(z)>0$ for all z (by standard Mill's ratio bounds).

Part (b) Fix $v\in\{\sigma_r^2,\sigma_o^2\}$; then $\partial\tau/\partial v = 1/(2\tau)$ and $\partial z_s/\partial v = -(\Delta_s)/(2\tau^3) = -z_s/(2\tau^2)$. Holding $a=\sigma_x^2$ fixed,

$$\frac{\partial B(s,+)}{\partial v} = a\left[-\frac{1}{\tau^2}\frac{\partial\tau}{\partial v}\rho(z_s)+\frac{1}{\tau}\rho'(z_s)\frac{\partial z_s}{\partial v}\right] = -\frac{a}{2\tau^3}\Big[\rho(z_s)+z_s\rho'(z_s)\Big].$$

Using $\rho'(z) = -\rho(z)(z+\rho(z))$ gives

$$\rho(z)+z\rho'(z) = \rho(z)\Big[1-z^2-z\rho(z)\Big] =: \rho(z)\,g(z),$$

so

$$\frac{\partial B(s,+)}{\partial v} = -\frac{a}{2\tau^3}\rho(z_s)\,g(z_s).$$

Since $a>0$, $\tau>0$, and $\rho(z_s)>0$, the sign is the opposite of $g(z_s)$. Note also that $g(0)=1$ and $\lim_{z\to+\infty} g(z) = -\infty$. Moreover, at any zero of g with $z>0$, we have $g'(z)<0$: this is because, using $\rho'(z) = -\rho(z)(z+\rho(z))$, we have $g'(z) = -2z-\rho(z)g(z)$, which is negative if $g(z)=0$. But then, there is a unique $z^*>0$ with $g(z^*)=0$ (numerically $z^*\simeq 0.84$). Therefore $g(z)>0$ for all $z\leq 0$ and for $0<z<z^*$, while $g(z)<0$ for $z>z^*$, which delivers the stated sign pattern.

Part (c) Holding $(\sigma_x^2,\sigma_r^2,\sigma_o^2)$ fixed, $B(s,+)=\frac{\sigma_x^2}{\tau}\rho(\Delta_s/\tau)$ and ρ is strictly decreasing, so

$$\frac{\partial B(s,+)}{\partial\Delta_s} = \frac{\sigma_x^2}{\tau^2}\rho'(z_s) \ < \ 0.$$

Part (d) Using $\lambda = \sigma_p^2/(\sigma_p^2+\sigma_\epsilon^2)$ and $\sigma_x^2 = \sigma_p^2\sigma_\epsilon^2/(\sigma_p^2+\sigma_\epsilon^2)$, note that $\Delta_s = \tilde{u}_x^s - \tilde{u}_r = \tilde{u}_x - \tilde{u}_r + \lambda(s-\tilde{u}_x)$ depends on $(\sigma_p^2,\sigma_\epsilon^2)$ only through λ, while σ_x^2 depends on $(\sigma_p^2,\sigma_\epsilon^2)$ through the usual posterior-variance formula. Hence the chain-rule decompositions displayed in the statement hold. Moreover,

$$\frac{\partial\sigma_x^2}{\partial\sigma_p^2}=\frac{\sigma_\epsilon^4}{(\sigma_p^2+\sigma_\epsilon^2)^2}>0,\qquad \frac{\partial\sigma_x^2}{\partial\sigma_\epsilon^2}=\frac{\sigma_p^4}{(\sigma_p^2+\sigma_\epsilon^2)^2}>0,$$

$$\frac{\partial\lambda}{\partial\sigma_p^2}=\frac{\sigma_\epsilon^2}{(\sigma_p^2+\sigma_\epsilon^2)^2}>0,\qquad \frac{\partial\lambda}{\partial\sigma_\epsilon^2}=-\frac{\sigma_p^2}{(\sigma_p^2+\sigma_\epsilon^2)^2}<0,$$

and $\partial\Delta_s/\partial\sigma_p^2=(s-\tilde{u}_x)\,\partial\lambda/\partial\sigma_p^2$, $\partial\Delta_s/\partial\sigma_\epsilon^2=(s-\tilde{u}_x)\,\partial\lambda/\partial\sigma_\epsilon^2$. Finally, part (a) implies $B_{\sigma_x^2}>0$ and part (c) implies $B_{\tilde{u}_x^s}=B_{\Delta_s}<0$. If $s\le\tilde{u}_x$, then $\partial\Delta_s/\partial\sigma_p^2\le 0$, so both channels weakly raise $B(s,+)$ and the inequality is strict; hence $\partial B(s,+)/\partial\sigma_p^2>0$. If $s\ge\tilde{u}_x$, then $\partial\Delta_s/\partial\sigma_\epsilon^2\le 0$, so both channels weakly raise $B(s,+)$ and the inequality is strict; hence $\partial B(s,+)/\partial\sigma_\epsilon^2>0$. In the complementary regions the two channels work in opposite directions, so the net sign is not determined in general. □

E.7 Proof of Proposition 15

Let $\delta:=u_x-u_r>0$ and recall that $s=u_x+\epsilon=u_r+\delta+\epsilon$ with $\epsilon\sim N(0,\sigma_\epsilon^2)$. Under $u_r=\tilde{u}_r=\tilde{u}_x$, the posterior mean after observing s is

$$\tilde{u}_x^s=u_r+\lambda(s-u_r)=u_r+\lambda(\delta+\epsilon),$$

and define $z_s:=(\tilde{u}_x^s-u_r)/\tau=\lambda(\delta+\epsilon)/\tau$, where $\tau:=\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}$. Let $\rho(z):=\frac{\phi(z)}{\Phi(z)}$.

Step 1 (closed form for C). For each realized s,

$$B(s,+)=\frac{\sigma_x^2}{\tau}\rho(z_s),\qquad B(s,-)=-\frac{\sigma_x^2}{\tau}\rho(-z_s).$$

Conditional on (u_x,u_r), the ordinal signal o depends only on v and is independent of ϵ (hence independent of s), with $p(\delta):=\Pr(o=+\mid u_x,u_r)=1$ if $\sigma_o^2=0$ and $p(\delta)=\Phi(\delta/\sigma_o)$ if $\sigma_o^2>0$. Therefore

$$\begin{aligned}C(u_x,u_r)&:=\mathbb{E}[B(s,o)\mid u_x,u_r]\\&=\mathbb{E}[p(\delta)B(s,+)+(1-p(\delta))B(s,-)\mid u_x,u_r]\\&=\frac{\sigma_x^2}{\tau}\,\mathbb{E}_\epsilon\left[p(\delta)\rho\left(\frac{\lambda(\delta+\epsilon)}{\tau}\right)-(1-p(\delta))\rho\left(-\frac{\lambda(\delta+\epsilon)}{\tau}\right)\right].\end{aligned}$$

Step 2 ($C(u_x,u_r)>0$). If $\sigma_o^2=0$, then $p(\delta)=1$ and the displayed expression reduces to $C(u_x,u_r)=\frac{\sigma_x^2}{\tau}\mathbb{E}[\rho(z_s)]>0$.

If $\sigma_o^2 > 0$, define

$$H_\delta(z) := p(\delta)\rho(z) - (1 - p(\delta))\rho(-z).$$

Using $\rho(z) = \phi(z)/\Phi(z)$ and $\rho(-z) = \phi(z)/(1 - \Phi(z))$, we have

$$H_\delta(z) = \phi(z)\left(\frac{p(\delta)}{\Phi(z)} - \frac{1 - p(\delta)}{1 - \Phi(z)}\right) = \phi(z)\,\frac{p(\delta) - \Phi(z)}{\Phi(z)\big(1 - \Phi(z)\big)}.$$

Hence $H_\delta(z)$ is strictly decreasing in z and has a unique zero at $z = \Phi^{-1}(p(\delta)) = \delta/\sigma_o$. Moreover, in the present setting $z_s = \lambda(\delta+\epsilon)/\tau$ is normally distributed with mean $\mathbb{E}[z_s] = \lambda\delta/\tau$, and since $\lambda < 1$ and $\tau > \sigma_o$ we have $\mathbb{E}[z_s] < \delta/\sigma_o$. Write $z_s = \mu_\delta + \sigma_z Y$ where $Y \sim N(0,1)$,$\mu_\delta := E[z_s] = \frac{\lambda\delta}{\tau}$, $\sigma_z := \sqrt{\mathbb{V}(z_s)} = \frac{\lambda\sigma_\epsilon}{\tau}$. Let $z_0 := \Phi^{-1}(p(\delta)) = \delta/\sigma_o$ and define the positive shift $c := z_0 - \mu_\delta > 0$. Since H_δ is strictly decreasing, we have the pointwise inequality $H_\delta(z_s) \geq H_\delta(z_s + c)$, hence

$$E_\epsilon[H_\delta(z_s)] \;\geq\; E_\epsilon[H_\delta(z_s + c)] = E[H_\delta(z_0 + \sigma_z Y)]\,.$$

By symmetry of Y,

$$E[H_\delta(z_0 + \sigma_z Y)] = \int_0^\infty \Big(H_\delta(z_0 + t) + H_\delta(z_0 - t)\Big)\,\frac{1}{\sigma_z}\phi\!\left(\frac{t}{\sigma_z}\right)\,dt.$$

Note that $H_\delta(z_0 + t) + H_\delta(z_0 - t) > 0$ for all $t > 0$.[49] But then, the integral is strictly positive. Therefore $E_\epsilon[H_\delta(z_s)] > 0$ and hence

$$C(u_x, u_r) = \frac{\sigma_x^2}{\tau}E_\epsilon[H_\delta(z_s)] > 0 \qquad \text{for all } \delta > 0.$$

Step 3 (comparative statics in σ_x^2). Fix (u_x, u_r), λ, σ_r^2, and σ_o^2, and write

$$a := \sigma_x^2 > 0, \qquad K := \sigma_r^2 + \sigma_o^2, \qquad \tau(a) := \sqrt{a + K}.$$

[49] This can be easily derived as follows. For every $p \in (0,1)$, let $H_p(z) := p\,\rho(z) - (1-p)\,\rho(-z) = \frac{\phi(z)\,(p - \Phi(z))}{\Phi(z)\,(1-\Phi(z))}$, and $z_0 := \Phi^{-1}(p)$. Then for every $t > 0$, we aim to show $H_p(z_0 - t) + H_p(z_0 + t) > 0$. To see why this holds, note that a standard property of the normal inverse Mills ratio is that ρ is *strictly convex* on $\mathbb{R}$. Fix $t > 0$. By strict convexity and Jensen's inequality, $\rho(z_0 - t) + \rho(z_0 + t) \;>\; 2\rho(z_0)$ and $\rho(-z_0 - t) + \rho(-z_0 + t) \;>\; 2\rho(-z_0)$, since both pairs $(z_0 - t, z_0 + t)$ and $(-z_0 - t, -z_0 + t)$ are symmetric around their midpoints. Multiply the first inequality by p and the second by $(1-p)$ and subtract to obtain $H_p(z_0 - t) + H_p(z_0 + t) = p\big(\rho(z_0 - t) + \rho(z_0 + t)\big) - (1 - p)\big(\rho(-z_0 - t) + \rho(-z_0 + t)\big) > \;2\big(p\rho(z_0) - (1-p)\rho(-z_0)\big)$. Finally, since $p = \Phi(z_0)$, $p\rho(z_0) = \Phi(z_0)\frac{\phi(z_0)}{\Phi(z_0)} = \phi(z_0)$, $(1-p)\rho(-z_0) = (1 - \Phi(z_0))\frac{\phi(z_0)}{\Phi(-z_0)} = \phi(z_0)$, so the right-hand side equals 0. Therefore $H_p(z_0 - t) + H_p(z_0 + t) > 0$ for every $t > 0$.

Recall the primitive identities

$$\lambda = \frac{\sigma_p^2}{\sigma_p^2 + \sigma_\epsilon^2}, \qquad \sigma_x^2 = \frac{\sigma_p^2 \sigma_\epsilon^2}{\sigma_p^2 + \sigma_\epsilon^2},$$

hence

$$\sigma_x^2 = \lambda \sigma_\epsilon^2.$$

Therefore, when we vary $a = \sigma_x^2$ holding λ fixed, we must have

$$\sigma_\epsilon^2 = \frac{a}{\lambda}, \qquad \text{so} \qquad \epsilon = \sqrt{\frac{a}{\lambda}}\, X, \quad X \sim N(0,1).$$

Let

$$p := p(\delta) = \begin{cases} 1, & \sigma_o^2 = 0, \\ \Phi(\delta/\sigma_o), & \sigma_o^2 > 0, \end{cases} \qquad H_\delta(z) := p\,\rho(z) - (1-p)\rho(-z).$$

By Step 1,

$$C(a) = \frac{a}{\tau(a)} E\left[H_\delta\left(\frac{\lambda\delta + \sqrt{\lambda a}\, X}{\tau(a)} \right) \right].$$

Define

$$Z_a := \frac{\lambda\delta + \sqrt{\lambda a}\, X}{\tau(a)}, \qquad m := \frac{\lambda\delta}{\tau(a)}, \qquad s := \frac{\sqrt{\lambda a}}{\tau(a)},$$

so that $Z_a = m + sX$ and

$$C(a) = \frac{a}{\tau(a)} E[H_\delta(Z_a)].$$

We use the standard facts that $\rho'(t) \in (-1, 0)$ for all t, $\rho''(t) > 0$ for all t, $\rho(t) \to 0$ as $t \to +\infty$, and $\rho(t) + t \to 0$ as $t \to -\infty$. In particular, ρ has at most linear growth, hence so does H_δ, while H_δ' is bounded.

Fix $a_0 > 0$ and let $I \subset (0, \infty)$ be a compact interval containing a_0. For $a \in I$, both Z_a and $\partial Z_a/\partial a$ are bounded by $C_I(1 + |X|)$ for some constant C_I. Since H_δ has at most linear growth and H_δ' is bounded, the maps

$$X \mapsto \frac{a}{\tau(a)} H_\delta(Z_a) \qquad \text{and} \qquad X \mapsto \frac{\partial}{\partial a}\left[\frac{a}{\tau(a)} H_\delta(Z_a) \right]$$

are dominated by an integrable envelope of the form $C_I(1 + |X|)$. Hence differentiation under the expectation is justified by dominated convergence.

Write $\tau := \tau(a)$ for brevity. Then

$$\frac{d}{da}\left(\frac{a}{\tau}\right) = \frac{a+2K}{2\tau^3},$$

and a direct calculation gives

$$\frac{\partial Z_a}{\partial a} = -\frac{\lambda\delta}{2\tau^3} + \frac{K\sqrt{\lambda}}{2\sqrt{a}\,\tau^3}\,X.$$

Therefore,

$$C'(a) = \frac{a+2K}{2\tau^3}E[H_\delta(Z_a)] - \frac{a\lambda\delta}{2\tau^4}E[H'_\delta(Z_a)] + \frac{K\sqrt{\lambda a}}{2\tau^4}E[XH'_\delta(Z_a)].$$

Now apply Stein's lemma to the standard normal X. Since $Z_a = m + sX$, we have

$$E[XH_\delta(Z_a)] = s\,E[H'_\delta(Z_a)],$$

and

$$E[(X^2-1)H_\delta(Z_a)] = s\,E[XH'_\delta(Z_a)].$$

Equivalently,

$$E[H'_\delta(Z_a)] = \frac{\tau}{\sqrt{\lambda a}}\,E[XH_\delta(Z_a)], \qquad E[XH'_\delta(Z_a)] = \frac{\tau}{\sqrt{\lambda a}}\,E[(X^2-1)H_\delta(Z_a)].$$

Substituting into the previous display yields

$$C'(a) = \frac{1}{2\tau^3}E\Big[\big(\tau^2 + KX^2 - \delta\sqrt{\lambda a}\,X\big)\,H_\delta(Z_a)\Big].$$

We now symmetrize the expectation. Let

$$W(x) := \tau^2 + Kx^2 - \delta\sqrt{\lambda a}\,x.$$

Using the standard normal density ϕ,

$$2\tau^3 C'(a) = \int_0^\infty \Big[W(x)H_\delta(m+sx) + W(-x)H_\delta(m-sx)\Big]\phi(x)\,dx.$$

Write

$$A(x) := \tau^2 + Kx^2 > 0, \qquad B(x) := \delta\sqrt{\lambda a}\,x > 0 \qquad (x > 0).$$

Then $W(x) = A(x) - B(x)$ and $W(-x) = A(x) + B(x)$, so the bracketed term equals

$$A(x)\Big(H_\delta(m+sx) + H_\delta(m-sx)\Big) + B(x)\Big(H_\delta(m-sx) - H_\delta(m+sx)\Big).$$

We claim that both parenthesized terms are strictly positive for every $x > 0$.

First, H_δ is strictly decreasing on $\mathbb{R}$. Indeed,

$$H'_\delta(z) = p\,\rho'(z) + (1-p)\rho'(-z) < 0 \qquad \text{for all } z,$$

because $\rho'(t) \in (-1, 0)$ for every t. Hence, for every $x > 0$,

$$H_\delta(m-sx) - H_\delta(m+sx) > 0.$$

It remains to show that

$$H_\delta(m+u) + H_\delta(m-u) > 0 \qquad \text{for every } u > 0.$$

If $\sigma_o^2 = 0$, then $p = 1$ and $H_\delta(z) = \rho(z) > 0$ for all z, so this is immediate.

Assume now that $\sigma_o^2 > 0$. Let

$$z_0 := \Phi^{-1}(p) = \frac{\delta}{\sigma_o}.$$

Since $\lambda \in (0,1)$ and $\tau > \sigma_o$, we have

$$m = \frac{\lambda\delta}{\tau} < \frac{\delta}{\sigma_o} = z_0.$$

By strict convexity of ρ,

$$\rho(z_0 - u) + \rho(z_0 + u) > 2\rho(z_0), \qquad \rho(-z_0 - u) + \rho(-z_0 + u) > 2\rho(-z_0)$$

for every $u > 0$. Multiply the first inequality by p and the second by $(1-p)$ and subtract:

$$H_\delta(z_0 - u) + H_\delta(z_0 + u) > 2\big(p\rho(z_0) - (1-p)\rho(-z_0)\big).$$

Since $p = \Phi(z_0)$,

$$p\rho(z_0) = \Phi(z_0)\frac{\phi(z_0)}{\Phi(z_0)} = \phi(z_0), \qquad (1-p)\rho(-z_0) = (1-\Phi(z_0))\frac{\phi(z_0)}{\Phi(-z_0)} = \phi(z_0),$$

so the right-hand side is 0. Therefore,

$$H_\delta(z_0 - u) + H_\delta(z_0 + u) > 0 \qquad \text{for every } u > 0.$$

Finally, since $m < z_0$ and H_δ is strictly decreasing,

$$H_\delta(m + u) > H_\delta(z_0 + u) \qquad \text{and} \qquad H_\delta(m - u) > H_\delta(z_0 - u),$$

hence

$$H_\delta(m + u) + H_\delta(m - u) > H_\delta(z_0 + u) + H_\delta(z_0 - u) > 0 \qquad \text{for every } u > 0.$$

Thus, for every $x > 0$, both parenthesized terms are strictly positive. Since also $A(x) > 0$, $B(x) > 0$, and $\phi(x) > 0$ on $(0, \infty)$, the integrand in the symmetrized formula is strictly positive for every $x > 0$. Therefore

$$C'(a) > 0.$$

Since $a = \sigma_x^2$, this proves that, holding λ fixed, $C(u_x, u_r)$ is strictly increasing in σ_x^2. This proves part (a).

Step 4 (comparative statics in δ and the hump shape). Write $C(\delta)$ for $C(u_x, u_r)$ as a function of δ (holding other parameters fixed). Differentiate the Step 1 formula with respect to δ (again exchanging differentiation and expectation by dominated convergence). If $\sigma_o^2 = 0$, then $p(\delta) \equiv 1$ and $z_s = \lambda(\delta + \epsilon)/\tau$ has $\partial z_s/\partial\delta = \lambda/\tau > 0$, so

$$C'(\delta) = \frac{\sigma_x^2}{\tau}\mathbb{E}[\rho'(z_s)]\,\frac{\lambda}{\tau} \;<\; 0,$$

because $\rho'(\cdot) < 0$. This proves part (c)(i).

If $\sigma_o^2 > 0$, then $p(\delta) = \Phi(\delta/\sigma_o)$ and $p'(\delta) = \phi(\delta/\sigma_o)/\sigma_o > 0$. A straightforward differentiation yields

$$C'(\delta) = \frac{\sigma_x^2}{\tau}\left[p'(\delta)\mathbb{E}(\rho(z_s) + \rho(-z_s)) + \frac{\lambda}{\tau}\mathbb{E}(p(\delta)\rho'(z_s) + (1 - p(\delta))\rho'(-z_s))\right].$$

Define the positive quantities

$$D(\delta) := \mathbb{E}(\rho(z_s) + \rho(-z_s)) > 0, \qquad N(\delta) := -\mathbb{E}(p(\delta)\rho'(z_s) + (1 - p(\delta))\rho'(-z_s)) > 0,$$

where $N(\delta) > 0$ since $\rho'(\cdot) < 0$ and $p(\delta) \in (0, 1)$. Then

$$C'(\delta) = \frac{\sigma_x^2}{\tau}\Big(p'(\delta)D(\delta) - \frac{\lambda}{\tau}N(\delta)\Big) = \frac{\sigma_x^2}{\tau}\, p'(\delta)D(\delta)\,\big(1 - K(\delta)\big), \tag{33}$$

where

$$K(\delta) := \frac{\lambda}{\tau}\frac{N(\delta)}{p'(\delta)D(\delta)}. \tag{34}$$

By dominated convergence, $D(\delta)$ and $N(\delta)$ are continuous in δ and finite for all $\delta > 0$. Moreover, as $\delta \downarrow 0$ we have $p(\delta) \to 1/2$ and $p'(\delta) \to \phi(0)/\sigma_o$, hence $K(\delta) \to K(0)$. To formally verify $K(0) < 1$, note that at $\delta = 0$, $z_s \sim \mathcal{N}(0, \sigma_z^2)$ is symmetric, yielding $D(0) = 2\mathbb{E}[\rho(z_s)]$ and $N(0) = -\mathbb{E}[\rho'(z_s)]$. Using $\rho'(z) = -\rho(z)(\rho(z) + z)$ and applying Stein's Lemma to the normal distribution yields $-\mathbb{E}[\rho'(z_s)] = \mathbb{E}[\rho(z_s)^2 + z_s\rho(z_s)] = \mathbb{E}[\rho(z_s)^2] + \sigma_z^2\mathbb{E}[\rho'(z_s)]$, which simplifies to $-\mathbb{E}[\rho'(z_s)] = \frac{1}{1+\sigma_z^2}\mathbb{E}[\rho(z_s)^2]$. Standard bounds on the inverse Mills ratio guarantee this evaluated ratio leaves $K(0)$ strictly bounded below 1 (converging to $\lambda < 1$ as $\sigma_o \to \infty$). Therefore, $1 - K(\delta) > 0$ and $C'(\delta) > 0$ for all sufficiently small $\delta > 0$.

As $\delta \to \infty$, $p'(\delta) = \phi(\delta/\sigma_o)/\sigma_o \to 0$ at the Gaussian rate $\exp(-\delta^2/(2\sigma_o^2))$. At the same time, $N(\delta) \to 0$ but at a strictly slower Gaussian rate because $z_s = \lambda(\delta + \epsilon)/\tau$ has mean $\lambda\delta/\tau$ with $\lambda/\tau < 1/\sigma_o$. Consequently $K(\delta) \to +\infty$, and hence $C'(\delta) < 0$ for all sufficiently large δ. Next, we prove that K is strictly increasing.

Claim 3 (Strict monotonicity of K). *Assume $\frac{\lambda}{\tau} < \frac{1}{\sigma_o}$. Then K is strictly increasing on $(0, \infty)$.*

Proof of Claim. We proceed in three steps.

Step 1 (preliminaries and regularity). The inverse Mills ratio satisfies the standard identities

$$\rho'(t) = \frac{d}{dt}\Big(\frac{\phi(t)}{\Phi(t)}\Big) = -\rho(t)\big(t + \rho(t)\big), \qquad \rho''(t) > 0 \quad \text{for all } t \in \mathbb{R},$$

so ρ is strictly decreasing and strictly convex. In particular, $\rho'(t) \in (-1, 0)$ for all t (hence $-\rho'(t) \in (0, 1)$), and $-\rho'$ is strictly decreasing. Define also

$$d(t) := \rho(t) + \rho(-t) = \frac{\phi(t)}{\Phi(t)(1 - \Phi(t))} > 0,$$

which is an even function, strictly increasing on $[0, \infty)$. Because ϵ is Gaussian and ρ, ρ' have at most polynomial growth on $\mathbb{R}$, dominated convergence implies D and N are C^1 on $(0, \infty)$ with $D(\delta) > 0$ and $N(\delta) > 0$ for all $\delta > 0$.

Step 2 (a convenient logarithmic derivative formula). Write $a := \lambda/\tau$. Since

$$p'(\delta) = \frac{1}{\sigma_o\sqrt{2\pi}} \exp\Big(-\frac{\delta^2}{2\sigma_o^2}\Big),$$

we can write

$$K(\delta) = a\,\sigma_o\sqrt{2\pi}\,\exp\Big(\frac{\delta^2}{2\sigma_o^2}\Big)\,\frac{N(\delta)}{D(\delta)}.$$

Hence K is strictly increasing if and only if

$$Q(\delta) := \exp\Big(\frac{\delta^2}{2\sigma_o^2}\Big)\,\frac{N(\delta)}{D(\delta)}$$

is strictly increasing. Differentiating $\log Q$ gives

$$\frac{Q'(\delta)}{Q(\delta)} = \frac{\delta}{\sigma_o^2} + \frac{N'(\delta)}{N(\delta)} - \frac{D'(\delta)}{D(\delta)}. \tag{35}$$

Step 3 (single-crossing comparison and positivity of (35)*).* Let $Z_\delta := z_s(\delta) = a(\delta+\epsilon)$, so $Z_\delta \sim \mathcal{N}(a\delta, a^2\sigma_\epsilon^2)$ is a strict normal location family in δ and therefore has the strict Monotone Likelihood Ratio Property (MLRP) in δ (equivalently, its kernel is strictly TP_2 in (z,δ)). Define the two (pointwise) nonnegative functions

$$h_+(z) := -\rho'(z) \in (0,1), \qquad h_-(z) := -\rho'(-z) \in (0,1),$$

where h_+ is strictly decreasing in z and h_- is strictly increasing in z. Also $d(z)$ is even and strictly increasing on $[0,\infty)$.

Introduce the (strictly positive) weighting coefficients

$$\alpha(\delta) := \frac{p(\delta)}{p'(\delta)}, \qquad \beta(\delta) := \frac{1-p(\delta)}{p'(\delta)}.$$

A direct calculation using $p(\delta) = \Phi(\delta/\sigma_o)$ and $p'(\delta) = \phi(\delta/\sigma_o)/\sigma_o$ yields

$$\alpha'(\delta) = 1 + \frac{\delta}{\sigma_o^2}\alpha(\delta), \qquad \beta'(\delta) = -1 + \frac{\delta}{\sigma_o^2}\beta(\delta). \tag{36}$$

Using α and β we can rewrite

$$\frac{N(\delta)}{p'(\delta)} = \alpha(\delta)\mathbb{E}[h_+(Z_\delta)] + \beta(\delta)\mathbb{E}[h_-(Z_\delta)].$$

Let

$$S(\delta) := \frac{N(\delta)}{p'(\delta)}, \qquad W(\delta) := D(\delta) = \mathbb{E}[d(Z_\delta)].$$

Then $Q(\delta) = e^{\delta^2/(2\sigma_o^2)} \frac{p'(\delta)S(\delta)}{W(\delta)} = e^{\delta^2/(2\sigma_o^2)} \frac{S(\delta)}{W(\delta)} \cdot p'(\delta)$, and (35) is equivalent to showing

$$\frac{d}{d\delta}\left(\frac{S(\delta)}{W(\delta)}\right) > -\frac{\delta}{\sigma_o^2}\frac{S(\delta)}{W(\delta)}. \tag{37}$$

Differentiate $S(\delta) = \alpha(\delta)\mathbb{E}[h_+(Z_\delta)] + \beta(\delta)\mathbb{E}[h_-(Z_\delta)]$ and use (36) to obtain

$$S'(\delta) = \big(\mathbb{E}[h_+(Z_\delta)] - \mathbb{E}[h_-(Z_\delta)]\big) + \frac{\delta}{\sigma_o^2}S(\delta) + \alpha(\delta)\frac{d}{d\delta}\mathbb{E}[h_+(Z_\delta)] + \beta(\delta)\frac{d}{d\delta}\mathbb{E}[h_-(Z_\delta)].$$

Therefore, after a routine quotient-rule rearrangement,

$$\frac{d}{d\delta}\left(\frac{S(\delta)}{W(\delta)}\right) + \frac{\delta}{\sigma_o^2}\frac{S(\delta)}{W(\delta)} = \frac{1}{W(\delta)^2}\,\Xi(\delta),$$

where

$$\Xi(\delta) := W(\delta)\big(\mathbb{E}[h_+(Z_\delta)]-\mathbb{E}[h_-(Z_\delta)]\big)+W(\delta)\Big(\alpha(\delta)\frac{d}{d\delta}\mathbb{E}[h_+(Z_\delta)]+\beta(\delta)\frac{d}{d\delta}\mathbb{E}[h_-(Z_\delta)]\Big)-S(\delta)W'(\delta).$$

We now show $\Xi(\delta) > 0$ for all $\delta > 0$.

By Gaussian differentiation under the integral sign, $\frac{d}{d\delta}\mathbb{E}[f(Z_\delta)] = a\,\mathbb{E}[f'(Z_\delta)]$ for any C^1 function f with suitable growth. Apply this to $f = h_+, h_-, d$ to write the last two terms in $\Xi(\delta)$ as expectations of h_+', h_-' and d'. Using that h_+ is strictly decreasing and d is strictly increasing on $[0,\infty)$, while h_- is strictly increasing and d is even, one verifies that the map

$$z \longmapsto \frac{h_+(z) - h_-(z)}{d(z)}$$

is strictly increasing on $\mathbb{R}$ (it is odd and increasing), and similarly the maps

$$z \longmapsto \frac{h_+'(z)}{d(z)} \quad \text{and} \quad z \longmapsto \frac{h_-'(z)}{d(z)}$$

are increasing on $\mathbb{R}$ (each is odd-increasing because h_+ is decreasing and convex while d is even and convex on $[0,\infty)$). Since Z_δ is a strict normal location family (hence strict MLRP/TP_2), Karlin's monotone (single-crossing) covariance theorem implies that, for any increasing ψ,

$$\mathrm{Cov}\big(\psi(Z_\delta),\, d(Z_\delta)\big) > 0 \qquad (\delta > 0),$$

and therefore each of the three bracketed contributions in $\Xi(\delta)$ is strictly positive once written as a $W(\delta)^2$-multiple of an appropriate covariance under the tilted measure proportional to $d(Z_\delta)$. Consequently $\Xi(\delta) > 0$ for all $\delta > 0$, which establishes (37) and hence $Q'(\delta) > 0$.

Finally, since $K(\delta)$ is a positive constant multiple of $Q(\delta)$, we conclude K is strictly increasing on $(0, \infty)$. ■

With this Claim, $K(\delta) = 1$ has a unique solution $\Delta^* > 0$, and (33) implies

$$C'(\delta) > 0 \text{ for } 0 < \delta < \Delta^*, \qquad C'(\delta) < 0 \text{ for } \delta > \Delta^*,$$

which establishes part (c)(ii). Moreover, dominated convergence and the IMR tail limits imply $C(\delta) \to 0$ as $\delta \downarrow 0$ and as $\delta \to \infty$.

Step 5 (effects of σ_r^2 and σ_o^2). Fix s and let $\Delta_s := \tilde{u}_x^s - u_r$ and $z_s := \Delta_s/\tau$ with $\tau := \sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}$. Let $v \in \{\sigma_r^2, \sigma_o^2\}$, so $\partial\tau/\partial v = 1/(2\tau)$ and v affects $\mathcal{B}(s, o)$ only through τ.

For $o = +$ we have $\mathcal{B}(s, +) = \frac{\sigma_x^2}{\tau}\rho(z_s)$, hence

$$\frac{\partial \mathcal{B}(s,+)}{\partial v} = \frac{\partial \mathcal{B}(s,+)}{\partial \tau}\frac{\partial \tau}{\partial v} = -\frac{\sigma_x^2}{2\tau^3}\Big(\rho(z_s) + z_s\rho'(z_s)\Big).$$

Using $\rho'(z) = -\rho(z)\big(z + \rho(z)\big)$, this becomes

$$\frac{\partial \mathcal{B}(s,+)}{\partial v} = \frac{\sigma_x^2}{2\tau^3}\rho(z_s)\Big(z_s^2 + z_s\rho(z_s) - 1\Big).$$

Define $g(z) := z^2 + z\rho(z) - 1$. Then $g(0) = -1$ and $g(z) \to +\infty$ as $z \to +\infty$. Moreover, for $z \geq 0$,

$$g'(z) = 2z + \rho(z) + z\rho'(z) \;\geq\; z + \rho(z) \;>\; 0,$$

since $\rho(z) > 0$ and $\rho'(z) \in (-1, 0)$. Hence there exists a unique $z^* > 0$ such that $g(z) < 0$ for $z \in [0, z^*)$ and $g(z) > 0$ for $z > z^*$. Therefore,

$$\frac{\partial \mathcal{B}(s,+)}{\partial v} < 0 \iff z_s < z^*, \qquad \frac{\partial \mathcal{B}(s,+)}{\partial v} > 0 \iff z_s > z^*.$$

(The analogous calculation for $o = -$ yields the same single-crossing structure in z_s.)

Now write $\delta := u_x - u_r$ and note that $z_s = \frac{\lambda}{\tau}(\delta + \varepsilon)$ is a normal location family in δ (with $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$). Because $\partial\mathcal{B}(s, o)/\partial v$ is a single-crossing function of z_s (with cutoff z^*) and the distribution of z_s shifts to the right in δ, $\delta \mapsto \mathbb{E}[\partial\mathcal{B}(s, o)/\partial v]$ is strictly increasing and crosses zero at most once. Moreover it is negative at $\delta = 0$ (since z_s is centered near 0 so $g(z_s) < 0$ with high probability) and positive for δ large (since $z_s \longrightarrow +\infty$ in probability so $g(z_s) > 0$ with high probability). Hence there exists a (unique) $\Delta_v > 0$ such that

$$\frac{\partial C}{\partial v} < 0 \text{ for } 0 < \delta < \Delta_v, \qquad \frac{\partial C}{\partial v} > 0 \text{ for } \delta > \Delta_v.$$

This proves part (b).

Step 6 (primitive variances). This is a direct chain-rule decomposition using $\sigma_x^2 = \sigma_p^2\sigma_\epsilon^2/(\sigma_p^2+\sigma_\epsilon^2)$ and $\lambda = \sigma_p^2/(\sigma_p^2+\sigma_\epsilon^2)$. The signs of $\partial\sigma_x^2/\partial\sigma_p^2$, $\partial\sigma_x^2/\partial\sigma_\epsilon^2$, $\partial\lambda/\partial\sigma_p^2$, and $\partial\lambda/\partial\sigma_\epsilon^2$ are immediate, and part (a) gives $C_{\sigma_x^2} > 0$ □

E.8 Proof of Proposition 16

Fix a dimension m and suppress the index. Write

$$x := u_{x,m}, \quad y := u_{y,m}, \quad z := u_{z,m},$$

and let the true values be (x_0, y_0, z_0) with $z_0 > x_0 > y_0$ (the case $x_0 > y_0 > z_0$ is symmetric by relabeling and sign reversal). By the maintained assumptions (independence across dimensions and the same prior and signal precisions for x, y, z in every dimension), posterior beliefs about $(u_{x,m}, u_{y,m}, u_{z,m})$ depend only on the signals in dimension m, so it suffices to work in this scalar model. For each pair $(i, j) \in \{(x, y), (z, x), (z, y)\}$ the ordinal signal is

$$I_{ij} := \mathbf{1}\{\, i > j + v_{ij} \,\}, \qquad v_{ij} \sim N(0, \sigma_o^2),$$

independent of each other and of everything else. Let $s := (s_x, s_y, s_z)$ and $o := (I_{xy}, I_{zx}, I_{zy})$. Define the ex ante average posterior mean difference

$$G(z_0) \;:=\; \mathbb{E}_{s,o|x_0,y_0,z_0}\big[\; \mathbb{E}[x - y \mid s, o] \;\big],$$

and recall that we call *range-contrast* the property $G'(z_0) > 0$ and *range-normalization* the property $G'(z_0) < 0$ (in the region $z_0 > x_0 > y_0$).

We now introduce two claims that we will use repeatedly.

Claim 4 (Increasing gap under left truncation). *Let $Y \sim N(m, \tau^2)$ and define $T(a) := \mathbb{E}[Y \mid Y < a]$. Then T is strictly increasing and $0 < T'(a) < 1$ for all a. Consequently,*

$$g(a) := a - T(a) \quad \textit{is strictly increasing in } a.$$

Proof of Claim. The truncated normal mean is $T(a) = m - \tau\,\lambda\big(\frac{a-m}{\tau}\big)$ where $\lambda(t) := \frac{\varphi(t)}{\Phi(t)}$ is the inverse Mills ratio. Differentiating yields

$$T'(a) = 1 - \lambda(t)\big(t + \lambda(t)\big), \qquad t = \frac{a-m}{\tau}.$$

Since $\lambda(t) > 0$ and $t + \lambda(t) > 0$ for all t, we have $T'(a) \in (0,1)$. Therefore $g'(a) = 1 - T'(a) \in (0,1)$, proving strict increase. ■

Claim 5 (Covariance with a monotone transformation)**.** *Let X be nondegenerate and integrable, and let h be integrable and weakly increasing, not a.s. constant. Then Cov$(h(X), X) > 0$.*

Proof of Claim. Let X' be an i.i.d. copy of X. Then

$$\mathrm{Cov}(h(X), X) = \frac{1}{2}\,\mathbb{E}\big[(h(X) - h(X'))(X - X')\big].$$

If h is increasing, the integrand is a.s. nonnegative and is strictly positive on a set of positive probability when h is not a.s. constant. ■

(a) Noiseless ordinals: $\sigma_o^2 = 0$ implies range-contrast.

Assume $\sigma_o^2 = 0$. Then each ordinal comparison is correct with probability 1, and (given $z_0 > x_0 > y_0$) the agent deterministically observes the ranking event

$$R := \{z > x > y\}.$$

Thus o is degenerate and we may write

$$M(s) \;:=\; \mathbb{E}[x - y \mid s, R], \qquad G(z_0) = \mathbb{E}_{s \mid x_0, y_0, z_0}[M(s)].$$

Step 1 (posterior under cardinals only). Ignoring the (now deterministic) ordinal information, the posterior given cardinals is independent and normal:

$$x \mid s_x \sim N(m_x, \tau^2), \quad y \mid s_y \sim N(m_y, \tau^2), \quad z \mid s_z \sim N(m_z, \tau^2),$$

where

$$\alpha := \frac{\sigma_p^2}{\sigma_p^2 + \sigma_\varepsilon^2}, \qquad m_j := (1-\alpha)\tilde{u} + \alpha s_j, \qquad \tau^2 := \frac{\sigma_p^2 \sigma_\varepsilon^2}{\sigma_p^2 + \sigma_\varepsilon^2}.$$

Step 2 (single-crossing in s_z). Condition on $R = \{z > x > y\}$. Integrating out z and y yields the conditional density of $X := x \mid (s, R)$:

$$f_{X|s,R}(x) \;\propto\; f_x(x)\, F_y(x)\, W_{m_z}(x), \tag{38}$$

where f_x is the $N(m_x, \tau^2)$ density, $F_y(x) := \P(y < x \mid s_y) = \Phi\big(\frac{x-m_y}{\tau}\big)$, and

$$W_{m_z}(x) := \P(z > x \mid s_z) = \Phi\Big(\frac{m_z - x}{\tau}\Big).$$

Only $W_{m_z}(x)$ depends on s_z (via m_z). The kernel $W_m(x) = \Phi((m - x)/\tau)$ is *log-supermodular* in (m, x):

$$\frac{\partial^2}{\partial m\, \partial x} \log W_m(x) = -\frac{1}{\tau^2}\, \lambda'\Big(\frac{m - x}{\tau}\Big) > 0,$$

because $\lambda'(t) = -\lambda(t)\big(t + \lambda(t)\big) < 0$ for all t. Hence the ratio $W_{m'}(x)/W_m(x)$ is increasing in x whenever $m' > m$, so the family $\{f_{X|s,R}(\cdot)\}_{m_z}$ has the monotone likelihood ratio property and therefore X is first-order stochastically increasing in m_z.

Next define, for fixed s_y,

$$g_y(x) \;:=\; x - \mathbb{E}[y \mid s_y,\; y < x].$$

By Claim 4, g_y is strictly increasing. Using iterated expectations,

$$M(s) = \mathbb{E}[x - y \mid s, R] = \mathbb{E}\big[g_y(X) \mid s, R\big]. \tag{39}$$

Since X is stochastically increasing in m_z and g_y is increasing, (39) implies that $M(s)$ is strictly increasing in m_z, hence strictly increasing in s_z (because m_z is strictly increasing in s_z). Define

$$H(t) := \mathbb{E}_{s_x, s_y | x_0, y_0}\big[M(s_x, s_y, t)\big].$$

Then H is strictly increasing.

Step 3 (average over s_z). Since $s_z \sim N(z_0, \sigma_\varepsilon^2)$,

$$G(z_0) = \mathbb{E}[H(s_z)].$$

Differentiating with respect to the location parameter z_0 yields

$$G'(z_0) = \frac{1}{\sigma_\varepsilon^2} \mathrm{Cov}\big(H(s_z), s_z\big).$$

By Claim 5, $\mathrm{Cov}(H(s_z), s_z) > 0$, so $G'(z_0) > 0$. Thus beliefs exhibit a range-contrast effect when

$\sigma_o^2 = 0$.

(b) Only ordinal information: $\sigma_o^2 > 0$ *and* $\sigma_\varepsilon^2 \to \infty$ *implies range-normalization.*

Assume $\sigma_o^2 > 0$ and $\sigma_\varepsilon^2 \to \infty$. Then cardinals are uninformative and the posterior depends only on the ordinal tournament $o = (I_{xy}, I_{zx}, I_{zy})$. Introduce the sign variables $T_{ij} := 2I_{ij} - 1 \in \{-1, +1\}$.

Claim 6 (Score representation with three alternatives). *There exists a constant* $\kappa > 0$ *(depending only on* σ_p^2 *and* σ_o^2*) such that for every realized ordinal tournament* o*,*

$$\mathbb{E}[x - y \mid o] = \kappa\big(S_x(o) - S_y(o)\big),$$

where $S_j(o) := \sum_{k \neq j} T_{jk}$ *is the win–loss score of alternative* j*. Equivalently,*

$$\mathbb{E}[x - y \mid o] = \kappa\big(2T_{xy} + T_{zy} - T_{zx}\big). \tag{40}$$

Proof of Claim. Because priors are i.i.d. and the probit likelihood depends only on pairwise differences, the posterior is exchangeable under relabeling of (x, y, z). Moreover, the reflection map $(x, y, z) \mapsto (2\tilde{u} - x, 2\tilde{u} - y, 2\tilde{u} - z)$ leaves the prior invariant and flips all pairwise comparisons (hence maps o to $-o$). Therefore

$$\mathbb{E}[x - \tilde{u} \mid -o] = -\mathbb{E}[x - \tilde{u} \mid o], \quad \text{and similarly for } y, z.$$

On three vertices, every tournament is either (i) cyclic (each alternative has one win and one loss) or (ii) transitive (one alternative has two wins, one has one win, one has zero wins). In a cyclic tournament, exchangeability implies $\mathbb{E}[x \mid o] = \mathbb{E}[y \mid o] = \mathbb{E}[z \mid o]$; since o is translation invariant and the prior mean is $\tilde{u}$, each equals $\tilde{u}$, so $\mathbb{E}[x - y \mid o] = 0$ and $S_x(o) - S_y(o) = 0$.

In a transitive tournament there is a unique middle alternative with score 0. Under reversal $-o$ the top and bottom swap but the middle remains the unique score-0 alternative; by the reflection symmetry above, the middle must satisfy $\mathbb{E}[\text{middle} \mid o] = \tilde{u}$. Since $\mathbb{E}[x \mid o] + \mathbb{E}[y \mid o] + \mathbb{E}[z \mid o] = 3\tilde{u}$, the top and bottom must be symmetric around $\tilde{u}$, i.e. $\tilde{u} \pm a$ for some $a > 0$ that does not depend on labels (by exchangeability). Then $\mathbb{E}[x - y \mid o]$ equals a when x is one rank above y and equals $2a$ when x is top and y is bottom. Since in these cases $S_x(o) - S_y(o)$ equals 2 and 4, respectively, (40) holds with $\kappa = a/2 > 0$. ∎

Taking expectations of (40) under the true values (x_0, y_0, z_0) gives

$$G(z_0) = \mathbb{E}_o[\mathbb{E}[x - y \mid o]] = \kappa\Big(2\,\mathbb{E}[T_{xy}] + \mathbb{E}[T_{zy}] - \mathbb{E}[T_{zx}]\Big).$$

Only the last two terms depend on z_0. Since $T_{ij} = +1$ with probability $\Phi(\frac{i_0 - j_0}{\sigma_o})$,

$$\mathbb{E}[T_{zx}] = 2\Phi\Big(\frac{z_0 - x_0}{\sigma_o}\Big) - 1, \qquad \mathbb{E}[T_{zy}] = 2\Phi\Big(\frac{z_0 - y_0}{\sigma_o}\Big) - 1.$$

Therefore,

$$G'(z_0) = \frac{2\kappa}{\sigma_o}\left(\varphi\Big(\frac{z_0 - y_0}{\sigma_o}\Big) - \varphi\Big(\frac{z_0 - x_0}{\sigma_o}\Big)\right).$$

Because $z_0 > x_0 > y_0$ implies $\frac{z_0 - y_0}{\sigma_o} > \frac{z_0 - x_0}{\sigma_o} > 0$ and φ is strictly decreasing on $(0, \infty)$, we have $G'(z_0) < 0$. Thus beliefs exhibit a range-normalization effect in the ordinal-only limit.

(c) Noisy ordinals and sufficiently noisy cardinals: local normalization near x_0 and contrast far away.

Fix $\sigma_o^2 > 0$. For each σ_ε^2, write $G_{\sigma_\varepsilon}(z_0)$ for the corresponding average posterior difference.

Step 1 (local normalization near x_0 for large σ_ε^2). By dominated convergence, as $\sigma_\varepsilon^2 \to \infty$ the cardinals become uninformative and $G_{\sigma_\varepsilon}(z_0) \to G_\infty(z_0)$ pointwise, where G_∞ is the ordinal-only function in part (b). Moreover, the likelihood in (z_0, s_z) is smooth and all relevant moments are finite, so one may differentiate under the expectation to obtain that $G'_{\sigma_\varepsilon}(z_0) \to G'_\infty(z_0)$ uniformly on compact z_0-intervals. Since part (b) gives $G'_\infty(x_0) < 0$, there exist $\delta > 0$ and $\bar{\sigma}_1^2$ such that for all $\sigma_\varepsilon^2 > \bar{\sigma}_1^2$,

$$G'_{\sigma_\varepsilon}(z_0) < 0 \qquad \text{for all } z_0 \in [x_0, x_0 + \delta]. \tag{41}$$

This is the desired (local) range-normalization effect when z_0 is close to the interval $[y_0, x_0]$.

Step 2 (contrast for z_0 sufficiently large when σ_ε^2 is large enough). Let E denote the event that the two comparisons involving z are observed as "z above both",

$$E := \{I_{zx} = 1,\ I_{zy} = 1\}.$$

Under the true values, $\P(E) = \Phi(\frac{z_0 - x_0}{\sigma_o})\Phi(\frac{z_0 - y_0}{\sigma_o}) \to 1$ as $z_0 \to \infty$, and $\P(E^c)$ and $\frac{d}{dz_0}\P(E)$ decay at Gaussian rates $\asymp \exp\big(-\frac{(z_0 - x_0)^2}{2\sigma_o^2}\big)$.

Decompose

$$G_{\sigma_\varepsilon}(z_0) = \P(E)\, G^+(z_0) + \P(E^c)\, G^-(z_0),$$

where $G^+(z_0) := \mathbb{E}[\mathbb{E}[x - y \mid s, o] \mid E]$ and $G^-(z_0) := \mathbb{E}[\mathbb{E}[x - y \mid s, o] \mid E^c]$. Differentiating,

$$G'_{\sigma_\varepsilon}(z_0) = \P'(E)\big(G^+(z_0) - G^-(z_0)\big) + \P(E)\,(G^+)'(z_0) + \P(E^c)\,(G^-)'(z_0). \tag{42}$$

We now lower bound $(G^+)'(z_0)$. Conditional on E, the variable s_z remains Gaussian $N(z_0, \sigma_\varepsilon^2)$

(because ordinal noise is independent of cardinal noise). Define

$$H^+(t) := \mathbb{E}\big[\mathbb{E}[x-y \mid s_x, s_y, t, o] \mid E\big],$$

so that $G^+(z_0) = \mathbb{E}[H^+(s_z)]$ with $s_z \sim N(z_0, \sigma_\varepsilon^2)$. An argument parallel to part (a) (based on the log-supermodularity of the probit kernel $\Phi((z-x)/\sigma_o)$ in (z, x) and closure under integration) implies that H^+ is strictly increasing and not a.s. constant.[50] Therefore, by Claim 5,

$$(G^+)'(z_0) = \frac{1}{\sigma_\varepsilon^2}\mathrm{Cov}\big(H^+(s_z), s_z\big) \;>\; 0. \tag{43}$$

Moreover, strict monotonicity implies that there exist $a < b$ and a constant $\Delta_H > 0$ such that $H^+(b) - H^+(a) \geq \Delta_H$. Using the representation $\mathrm{Cov}(h(X), X) = \frac{1}{2}\mathbb{E}[(h(X) - h(X'))(X - X')]$ and restricting to the event $\{X \geq b,\ X' \leq a\}$ yields the (lower) bound

$$\mathrm{Cov}(H^+(s_z), s_z) \;\geq\; \frac{1}{2}\,\Delta_H\,(b-a)\,\P(s_z \geq b)\,\P(s_z \leq a).$$

Since $\P(s_z \geq b) \to 1$ and $\P(s_z \leq a) = \Phi(\frac{a - z_0}{\sigma_\varepsilon})$,

$$(G^+)'(z_0) \;\geq\; \frac{c_0}{\sigma_\varepsilon^2}\,\Phi\Big(\frac{a - z_0}{\sigma_\varepsilon}\Big) \qquad \text{for some } c_0 > 0 \text{ and all large } z_0. \tag{44}$$

Next, the two remaining terms in (42) are of order $\exp\big(-\frac{(z_0 - x_0)^2}{2\sigma_o^2}\big)$, because $\P'(E)$ and $\P(E^c)$ have this Gaussian tail rate and $\big|G^+(z_0) - G^-(z_0)\big|$, $|(G^-)'(z_0)|$ grow at most polynomially in z_0 (posterior means have finite moments under the Gaussian prior and likelihood). In particular, there exists $C > 0$ and $k \geq 0$ such that for all large z_0,

$$\Big|\P'(E)\big(G^+(z_0) - G^-(z_0)\big) + \P(E^c)\,(G^-)'(z_0)\Big| \;\leq\; C(1 + z_0^k)\exp\Big(-\frac{(z_0 - x_0)^2}{2\sigma_o^2}\Big). \tag{45}$$

Combining (42), (44), and (45), we obtain that if $\sigma_\varepsilon^2 > \sigma_o^2$ then the positive term $(G^+)'(z_0)$ eventually dominates the two remaining terms, because $\Phi(\frac{a - z_0}{\sigma_\varepsilon})$ decays at the slower Gaussian rate $\exp\big(-\frac{z_0^2}{2\sigma_\varepsilon^2}\big)$ while (45) decays at rate $\exp\big(-\frac{z_0^2}{2\sigma_o^2}\big)$. Hence, for each $\sigma_\varepsilon^2 > \sigma_o^2$ there exists $Z(\sigma_\varepsilon) > x_0$ such that

$$G'_{\sigma_\varepsilon}(z_0) > 0 \qquad \text{for all } z_0 \geq Z(\sigma_\varepsilon). \tag{46}$$

[50]Formally, conditional on E the joint posterior density is proportional to the product of normal terms from cardinals and probit likelihood terms $\Phi((z-x)/\sigma_o)\Phi((z-y)/\sigma_o)$; the cross-partial $\frac{\partial^2}{\partial z\,\partial x}\log\Phi((z-x)/\sigma_o) > 0$ yields a monotone likelihood ratio shift in x as the posterior mean of z increases, and Claim 4 then implies that the posterior mean of $x - y$ increases.

Step 3 (single crossing). Let $\bar{\sigma}^2 := \max\{\bar{\sigma}_1^2, \sigma_o^2\}$. If $\sigma_\varepsilon^2 > \bar{\sigma}^2$, then (41) gives $G'_{\sigma_\varepsilon}(z_0) < 0$ for $z_0 \in [x_0, x_0 + \delta]$ (local range-normalization), while (46) gives $G'_{\sigma_\varepsilon}(z_0) > 0$ for all sufficiently large z_0 (range-contrast far away). By continuity of G'_{σ_ε} in z_0, there is at least one cutoff at which the derivative changes sign, i.e. the comparative statics exhibit the stated normalization–then–contrast pattern. (As $\sigma_\varepsilon^2 \to \infty$, the cutoff $Z(\sigma_\varepsilon)$ diverges, which is consistent with part (b).)

This completes the proof of the claim when $u_{z,m} > u_{x,m} > u_{y,m}$. The proof of the opposite case is specular. □

E.9 Proof of Proposition 17

For any information set I, let

$$U(x) := \sum_{i=1}^{n} u_{x,i}.$$

By cautious representation,

$$V(x \mid I) = E\left[-\frac{e^{-\alpha U(x)}}{\alpha}\,\middle|\, I\right] = -\frac{1}{\alpha} E\left[e^{-\alpha \sum_{i=1}^{n} u_{x,i}}\,\middle|\, I\right].$$

Because the information sets considered in the paper preserve conditional independence across dimensions,

$$E\left[e^{-\alpha \sum_{i=1}^{n} u_{x,i}}\,\middle|\, I\right] = \prod_{i=1}^{n} E[e^{-\alpha u_{x,i}} \mid I]\,.$$

Hence

$$V(x \mid I) = -\frac{1}{\alpha} \prod_{i=1}^{n} E[e^{-\alpha u_{x,i}} \mid I]\,.$$

Now define

$$g_\alpha(z) := -\frac{1}{\alpha} \ln(-\alpha z),$$

on its natural domain $\{z : -\alpha z > 0\}$. Since

$$g'_\alpha(z) = -\frac{1}{\alpha z} > 0$$

throughout this domain, g_α is strictly increasing. Therefore $g_\alpha \circ V$ represents the same preference relation as V. Using the previous display,

$$g_\alpha(V(x \mid I)) = -\frac{1}{\alpha} \ln\left(\prod_{i=1}^{n} E[e^{-\alpha u_{x,i}} \mid I]\right) = \sum_{i=1}^{n} \left(-\frac{1}{\alpha} \ln E[e^{-\alpha u_{x,i}} \mid I]\right).$$

Thus $\succsim$ is represented by

$$\bar{V}(x \mid I) := \sum_{i=1}^{n} CE(u_{x,i} \mid I), \qquad CE(u_{x,i} \mid I) := -\frac{1}{\alpha} \ln E[e^{-\alpha u_{x,i}} \mid I].$$

Next, if $u_x \mid s \sim N(\tilde{u}_x^s, \sigma_x^2)$, then the moment generating function of the normal implies

$$E[e^{-\alpha u_x} \mid s] = \exp\left(-\alpha \tilde{u}_x^s + \frac{\alpha^2 \sigma_x^2}{2}\right).$$

Therefore,

$$CE(u_x \mid s) = -\frac{1}{\alpha} \ln E[e^{-\alpha u_x} \mid s] = -\frac{1}{\alpha}\left(-\alpha \tilde{u}_x^s + \frac{\alpha^2 \sigma_x^2}{2}\right) = \tilde{u}_x^s - \frac{\alpha \sigma_x^2}{2}.$$

For the ordinal signal, let

$$X := u_x \mid s \sim N(\mu_x, \sigma_x^2), \qquad R := u_r \sim N(\mu_r, \sigma_r^2), \qquad Y := X - R - \nu,$$

where $\mu_x = \tilde{u}_x^s$, $\mu_r = \tilde{u}_r$, and $\nu \sim N(0, \sigma_o^2)$ is independent of (X, R). Then

$$o = + \iff Y \geq 0, \qquad o = - \iff Y < 0.$$

Moreover,

$$Y \sim N(\mu_Y, \tau^2), \qquad \mu_Y = \mu_x - \mu_r, \qquad \tau^2 = \sigma_x^2 + \sigma_r^2 + \sigma_o^2,$$

and

$$\mathrm{Cov}(X, Y) = \sigma_x^2.$$

We first treat the case $o = +$. Since (X, Y) is jointly normal,

$$X \mid Y = y \sim N\left(\mu_x + \frac{\sigma_x^2}{\tau^2}(y - \mu_Y),\ \sigma_x^2\left(1 - \frac{\sigma_x^2}{\tau^2}\right)\right).$$

Hence

$$E[e^{-\alpha X} \mid Y = y] = \exp\left(-\alpha\mu_x - \alpha\frac{\sigma_x^2}{\tau^2}(y - \mu_Y) + \frac{\alpha^2}{2}\sigma_x^2\left(1 - \frac{\sigma_x^2}{\tau^2}\right)\right).$$

Therefore

$$E[e^{-\alpha X}\mathbf{1}_{\{Y \geq 0\}}] = C\,E[e^{tY}\mathbf{1}_{\{Y \geq 0\}}],$$

where

$$C = \exp\left(-\alpha\mu_x + \alpha\frac{\sigma_x^2}{\tau^2}\mu_Y + \frac{\alpha^2}{2}\sigma_x^2\left(1 - \frac{\sigma_x^2}{\tau^2}\right)\right), \qquad t = -\alpha\frac{\sigma_x^2}{\tau^2}.$$

For a normal random variable $Y \sim N(\mu_Y, \tau^2)$, completing the square gives

$$E[e^{tY}\mathbf{1}_{\{Y\geq 0\}}] = \exp\left(t\mu_Y + \frac{t^2\tau^2}{2}\right)\Phi\left(\frac{\mu_Y + t\tau^2}{\tau}\right).$$

Applying this with the above value of t yields

$$E[e^{-\alpha X}\mathbf{1}_{\{Y\geq 0\}}] = \exp\left(-\alpha\mu_x + \frac{\alpha^2\sigma_x^2}{2}\right)\Phi\left(\frac{\mu_Y - \alpha\sigma_x^2}{\tau}\right).$$

Since $P(Y \geq 0) = \Phi(\mu_Y/\tau)$,

$$E[e^{-\alpha X} \mid Y \geq 0] = \exp\left(-\alpha\mu_x + \frac{\alpha^2\sigma_x^2}{2}\right)\frac{\Phi\left(\frac{\mu_Y - \alpha\sigma_x^2}{\tau}\right)}{\Phi\left(\frac{\mu_Y}{\tau}\right)}.$$

Taking $-\frac{1}{\alpha}\log$ on both sides,

$$CE(X \mid Y \geq 0) = \mu_x - \frac{\alpha\sigma_x^2}{2} + \frac{1}{\alpha}\left[\ln\Phi\left(\frac{\mu_Y}{\tau}\right) - \ln\Phi\left(\frac{\mu_Y - \alpha\sigma_x^2}{\tau}\right)\right].$$

Now define

$$m := \frac{\mu_x - \mu_r}{\tau}, \qquad k := \frac{\sigma_x^2}{\tau}.$$

Since $\mu_Y = \mu_x - \mu_r$, this becomes

$$CE(x \mid s, +) = \tilde{u}_x^s - \frac{\alpha\sigma_x^2}{2} + \frac{1}{\alpha}\big(\ln\Phi(m) - \ln\Phi(m - \alpha k)\big).$$

The case $o = -$ is analogous. Using

$$E[e^{tY}\mathbf{1}_{\{Y<0\}}] = \exp\left(t\mu_Y + \frac{t^2\tau^2}{2}\right)\Phi\left(\frac{-\mu_Y - t\tau^2}{\tau}\right),$$

with the same $t = -\alpha\sigma_x^2/\tau^2$, we obtain

$$E[e^{-\alpha X} \mid Y < 0] = \exp\left(-\alpha\mu_x + \frac{\alpha^2\sigma_x^2}{2}\right)\frac{\Phi\left(\frac{-\mu_Y + \alpha\sigma_x^2}{\tau}\right)}{\Phi\left(\frac{-\mu_Y}{\tau}\right)}.$$

Therefore

$$CE(x \mid s, -) = \tilde{u}_x^s - \frac{\alpha\sigma_x^2}{2} - \frac{1}{\alpha}\big(\ln\Phi(-m+\alpha k) - \ln\Phi(-m)\big).$$

Combining the two cases yields the stated formula. □

E.10 Proof of Proposition 18

For each $z \in \{x, m\}$, let

$$\Delta_z := \sigma_z^2 - \hat{\sigma}_z^2, \qquad \tau_z := \sqrt{\sigma_z^2 + \hat{\sigma}_z^2},$$

and define

$$\Gamma_z(a) := a - \frac{\alpha}{2}\Delta_z + \frac{1}{\alpha}\log\frac{\Phi\left(\dfrac{a+\alpha\hat{\sigma}_z^2}{\tau_z}\right)}{\Phi\left(\dfrac{a-\alpha\sigma_z^2}{\tau_z}\right)},$$

$$\Lambda_z(a) := a + \frac{\alpha}{2}\Delta_z + \frac{1}{\alpha}\log\frac{\Phi\left(\dfrac{a+\alpha\sigma_z^2}{\tau_z}\right)}{\Phi\left(\dfrac{a-\alpha\hat{\sigma}_z^2}{\tau_z}\right)}.$$

A direct rearrangement of Eqs. (21)–(23) yields

$$\Lambda_m(p_{\mathrm{WTP}}) = \Gamma_x(u_x), \tag{47}$$

$$\Gamma_m(p_{\mathrm{Ch}}) = \Gamma_x(u_x), \tag{48}$$

and

$$\Gamma_m(p_{\mathrm{WTA}}) = \Lambda_x(u_x). \tag{49}$$

We first prove part 1. Let

$$\lambda(t) := \frac{\phi(t)}{\Phi(t)}$$

denote the inverse Mills ratio. Differentiating gives

$$\Gamma_z'(a) = 1 + \frac{1}{\alpha\tau_z}\left[\lambda\left(\frac{a+\alpha\hat{\sigma}_z^2}{\tau_z}\right) - \lambda\left(\frac{a-\alpha\sigma_z^2}{\tau_z}\right)\right].$$

The two arguments differ by

$$\frac{(a+\alpha\hat{\sigma}_z^2) - (a-\alpha\sigma_z^2)}{\tau_z} = \alpha\tau_z.$$

Hence, by the mean value theorem, there exists ξ_a between those two arguments such that

$$\Gamma_z'(a) = 1 + \lambda'(\xi_a).$$

For the normal inverse Mills ratio,

$$-1 < \lambda'(t) < 0 \qquad \text{for all } t \in \mathbb{R},$$

because, for $Z \sim N(0,1)$,

$$1 + \lambda'(t) = \mathrm{Var}(Z \mid Z > -t) \in (0,1).$$

Therefore

$$0 < \Gamma_z'(a) < 1 \qquad \text{for all } a \in \mathbb{R}.$$

Exactly the same argument gives

$$0 < \Lambda_z'(a) < 1 \qquad \text{for all } a \in \mathbb{R}.$$

So Γ_z and Λ_z are continuous and strictly increasing.

Next, as $a \to +\infty$, the logarithmic terms converge to 0, so

$$\Gamma_z(a) \to +\infty, \qquad \Lambda_z(a) \to +\infty.$$

As $a \to -\infty$, use the standard Mills-ratio asymptotic

$$\log \Phi(x+c) - \log \Phi(x) = -cx - \frac{c^2}{2} + o(1) \qquad (x \to -\infty).$$

Applying this to Γ_z with

$$x = \frac{a - \alpha\sigma_z^2}{\tau_z}, \qquad c = \alpha\tau_z,$$

gives

$$\frac{1}{\alpha} \log \frac{\Phi\left(\dfrac{a + \alpha\hat{\sigma}_z^2}{\tau_z}\right)}{\Phi\left(\dfrac{a - \alpha\sigma_z^2}{\tau_z}\right)} = -a + \frac{\alpha}{2}\Delta_z + o(1),$$

hence

$$\Gamma_z(a) \to 0 \qquad (a \to -\infty).$$

Applying the same asymptotic to Λ_z with

$$x = \frac{a - \alpha\hat{\sigma}_z^2}{\tau_z}, \qquad c = \alpha\tau_z,$$

gives

$$\frac{1}{\alpha}\log\frac{\Phi\left(\dfrac{a + \alpha\sigma_z^2}{\tau_z}\right)}{\Phi\left(\dfrac{a - \alpha\hat{\sigma}_z^2}{\tau_z}\right)} = -a - \frac{\alpha}{2}\Delta_z + o(1),$$

hence

$$\Lambda_z(a) \to 0 \qquad (a \to -\infty).$$

Thus Γ_z and Λ_z are continuous strictly increasing bijections from $\mathbb{R}$ onto $(0, \infty)$. Since $\Gamma_x(u_x), \Lambda_x(u_x) \in (0, \infty)$, (47)–(49) each admit a unique solution. This proves part 1.

We now prove part 2. First note that, by direct inspection of the definitions,

$$\Lambda_z(a) = \Gamma_z(a + \alpha\Delta_z) \qquad \text{for all } a \in \mathbb{R} \text{ and } z \in \{x, m\}. \tag{50}$$

Indeed, substituting $a + \alpha\Delta_z$ into Γ_z transforms $a + \alpha\hat{\sigma}_z^2$ into $a + \alpha\sigma_z^2$ and $a - \alpha\sigma_z^2$ into $a - \alpha\hat{\sigma}_z^2$.

Because the endowment comes with an additional cardinal signal of noise $\sigma_e^2 > 0$, the posterior variance is strictly smaller when the object is the endowment. Hence, whenever $\sigma_x^2 > 0$,

$$\hat{\sigma}_x^2 < \sigma_x^2,$$

so $\Delta_x > 0$.

Using (50) in (49),

$$\Gamma_m(p_{\text{WTA}}) = \Lambda_x(u_x) = \Gamma_x(u_x + \alpha\Delta_x).$$

Combining this with (48),

$$\Gamma_m(p_{\text{Ch}}) = \Gamma_x(u_x),$$

and using that Γ_x is strictly increasing and $\alpha\Delta_x > 0$, we obtain

$$\Gamma_x(u_x + \alpha\Delta_x) > \Gamma_x(u_x).$$

Since Γ_m is strictly increasing, it follows that

$$p_{\text{WTA}} > p_{\text{Ch}}.$$

Next, applying (50) to (47),

$$\Gamma_m(p_{\text{WTP}} + \alpha\Delta_m) = \Lambda_m(p_{\text{WTP}}) = \Gamma_x(u_x) = \Gamma_m(p_{\text{Ch}}).$$

By the injectivity of Γ_m,

$$p_{\text{Ch}} = p_{\text{WTP}} + \alpha\Delta_m.$$

Therefore:

- if $\sigma_m^2 > 0$, then by the same logic the endowed-money posterior variance is strictly smaller than the non-endowed one, so $\hat{\sigma}_m^2 < \sigma_m^2$, hence $\Delta_m > 0$, and therefore

$$p_{\text{Ch}} > p_{\text{WTP}};$$

- if $\sigma_m^2 = 0$, then necessarily $\hat{\sigma}_m^2 = 0$, so $\Delta_m = 0$, and therefore

$$p_{\text{Ch}} = p_{\text{WTP}}.$$

Combining these conclusions with $p_{\text{WTA}} > p_{\text{Ch}}$, we obtain

$$p_{\text{WTA}} > p_{\text{Ch}} > p_{\text{WTP}} \qquad \text{if } \sigma_m^2 > 0,$$

and

$$p_{\text{WTA}} > p_{\text{Ch}} = p_{\text{WTP}} \qquad \text{if } \sigma_m^2 = 0.$$

In particular, in both cases,

$$p_{\text{WTA}} > p_{\text{WTP}}.$$

This proves part 2.

F Additional Results on Convenient Approximations

F.1 Approximation of Inverse Mills Ratio

We compare the approximation $\psi^a(\cdot)$, as defined in (7), with the true inverse Mill's ratio (IMR) function $\psi(\cdot)$, as defined in (5). In order to do so, we simulate the posterior mean $\mathbb{E}[\tilde{u}_x^{s,o}]$ (averaged over signal realizations) when the agent receives an ordinal signal under both models.

Formally, we compute

$$\text{Full Model:} \qquad \frac{1}{N}\sum_{i=1}^{N}\left[\lambda s + (1-\lambda)\tilde{u}_x + \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}\,\psi\left(\frac{\lambda s + (1-\lambda)\tilde{u}_x - \tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}, o\right)\right]$$

$$\text{Approximation:} \qquad \frac{1}{N}\sum_{i=1}^{N}\left[\lambda s + (1-\lambda)\tilde{u}_x + \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}\,\psi^a\left(\frac{\lambda s + (1-\lambda)\tilde{u}_x - \tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}, o\right)\right]$$

Where cardinal and ordinal errors are drawn independently for $i = 1, 2, ..., N$. We let $N = 10,000$ and fix parameters $\tilde{u}_x = \tilde{u}_r = u_r = 0$ and $\sigma_p = \sigma_r = 1$. We compare the simulation average across $u_x \in [-2, 2]$ for all combinations of $\sigma_\epsilon \in \left\{\frac{1}{2}, 1, 2\right\}$ and $\sigma_0 \in \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$. Figure 8 displays the simulated means under the full and approximated model. The approximation $\psi^a(\cdot)$ performs well, and is virtually indistinguishable from the full model.

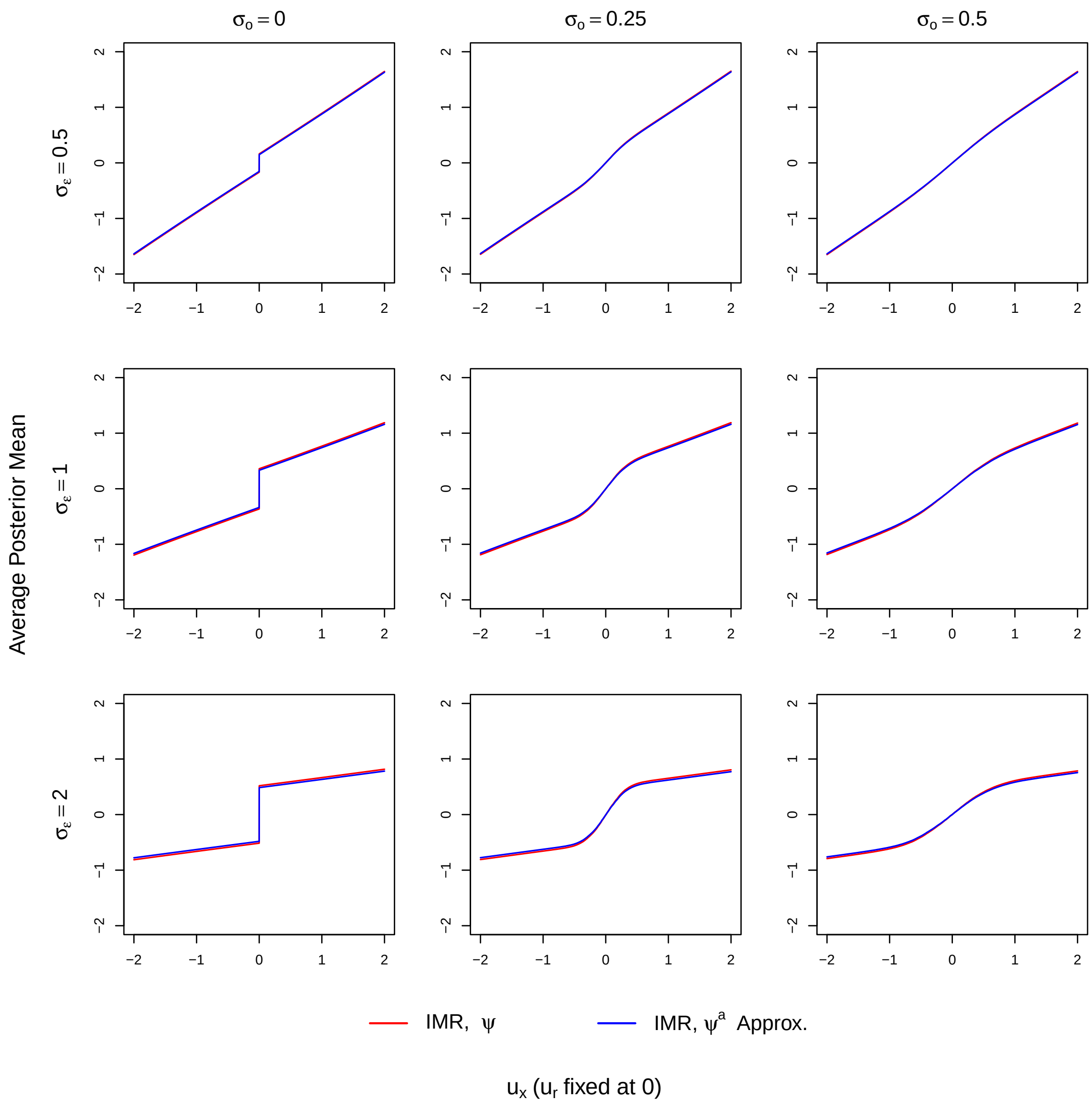

Figure 8: Simulation of posterior mean using IMR and its approximation $\psi^a(\cdot)$, averaged in both cases over 10,000 error draws. Prior and reference point parameters are set to $\tilde{u}_x = \tilde{u}_r = u_r = 0$ and utility is given by $u_x = x$, which we vary between [-2,2].

F.2 Approximation of Average Inverse Mills Ratio

We simulate the average posterior mean $\mathbb{E}[\tilde{u}_x^{s,o}]$ for the full model as in Appendix F.1 by averaging across 10,000 realizations of cardinal and ordinal signals. We compare this simulation average to the approximation of (8) (computed once per u_x) for all combinations of $\sigma_\epsilon \in \left\{\frac{1}{2}, 1, 2\right\}$ and $\sigma_0 \in \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$. In all simulations, we set $\tilde{u}_x = \tilde{u}_r = u_r = 0$ and $\sigma_p = \sigma_r = 1$. Figures 9 illustrates the simulation results. We find that the approximation computed using $\overline{\psi}^a(\cdot)$ closely tracks the true average.

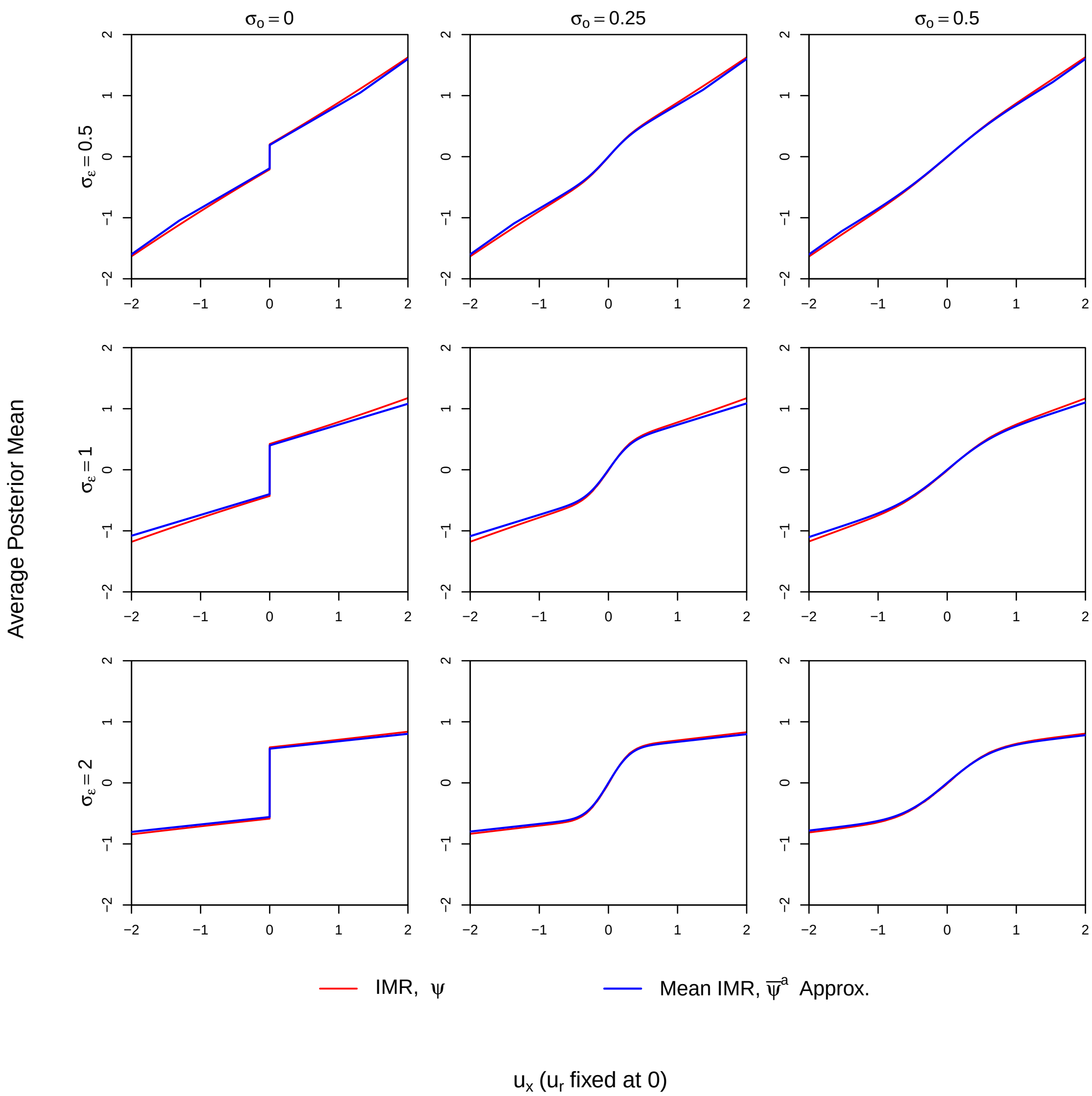

Figure 9: Simulation of Posterior Mean using IMR (average over 10,000 error draws) and its approximation $\overline{\psi}^a(\cdot)$. Prior and reference point parameters are set to $\tilde{u}_x = \tilde{u}_r = u_r = 0$ and utility is given by $u_x = x$, which we vary between [-2,2].

F.3 Approximation of Bayesian Continuous Action Choice Model

We use simulations to assess the quality of the quasi-Bayesian model of continuous action choice in Section 4.7. The DM evaluates the utilities of different levels of an action, a. Given the additive separability of our model, we here focus on a single dimension.

Bayesian model. The DM has a prior belief about the dimension-specific utility of action a, $u_a|a \sim \mathcal{N}(\tilde{u}_a, \sigma_p^2)$, where the prior mean may depend on the action. As laid out in Section 4.7, the DM receives two types of signals. First, for each action, a cardinal signal with a noise term that is independent across a.

$$s^a_{\text{Bayesian}} \sim \mathcal{N}(u_a, \sigma^2_{\epsilon,\text{Bayesian}}).$$

Second, the DM receives noiseless ordinal signals that tell him, for each action and associated utility outcome u_a, whether $u_a \gtreqless u_{a'}$ for all u_a, $u_{a'}$. In other words, the DM understands monotonicity not just with respect to the reference point but also with respect to the outcomes induced by his actions. For example, he may not know his earnings utility from working 8 hours but he does not that his earnings utility from working 8 hours is higher than that from working 7 hours. These ordinal signals ensure that the posterior mean about u_a is monotonic in the action. However, the interdependent truncations arising from the many ordinal signals make the Bayesian posterior mean intractable.

Computationally, we explicitly enforce knowledge of monotonicity through a constrained posterior estimation process. Specifically, we employ Hamiltonian Monte Carlo No-U-Turn Sampler (NUTS sampler) from the `PyMC` package with an ordered transformation constraining the parameter space, resulting in a posterior distribution that is monotonic but analytically not tractable.

Quasi-Bayesian model. We approximate this model by making use of the idea that – just like independent noise terms in combination with ordinal signals ensure monotonic posteriors – correlated noise terms also ensure monotonicity. Suppose there is a single error draw $\epsilon_0 \sim \mathcal{N}(0, \sigma^2_{\epsilon,\text{quasi-Bayesian}})$, and that the DM receives signals

$$s^a_{\text{quasi-Bayesian}} = u_a + \epsilon_0.$$

Further suppose that the DM updates his prior belief from these cardinal signals by fully ignoring the correlation among them. Then, the quasi-Bayesian posterior mean is given by

$$\mathbb{E}[\, u_a \,|\, s^a_{\text{quasi-Bayesian}} \,] = \lambda \, s^a_{\text{quasi-Bayesian}} + (1-\lambda)\, \tilde{u}_a \tag{51}$$

where λ is the usual shrinkage weight.

The thought behind our approximated model is that, by picking a noise variance, the quasi-Bayesian posterior mean from the model with perfectly correlated noise terms approximates the Bayesian posterior of the model with independent noise and ordinal signals. We thus pick the variance $\sigma^2_{\epsilon,\text{quasi-Bayesian}}$ that minimizes the distance between the two posterior means through nested interval optimization. Specifically, given the prior mean, prior variance and the Bayesian DM's signal variance, we identify the optimal quasi-Bayesian variance that minimizes the Mean Squared Error (MSE) between the posterior means of the quasi-Bayesian and constrained Bayesian models.

Simulation. We simulate both models over a range of parameter combinations using Python's `PyMC` package with the Hamiltonian Monte Carlo No-U-Turn Sampler (NUTS). Figure 10 displays the average posterior mean as a function of a for both the Bayesian and quasi-Bayesian DMs. The Bayesian posterior mean is computed by explicitly enforcing monotonicity via an ordered transformation, whereas the quasi-Bayesian posterior mean utilizes the analytical formula given by equation (51). The results indicate that the quasi-Bayesian approximation closely replicates the average posterior means produced by the fully Bayesian model, with deviations observed near the boundaries of the simulated range.

Quality of approximation with a reference point. The simulations above suggest that the posterior means of the two models are close to each other when there is no reference point. We now extend the analysis to a reference point, r. Specifically, the DM now additionally receives ordinal signals o_a that indicate whether $u_a \gtrless u_r$. Because this introduces an additional layer of truncation in addition to the monotonicity constraints among the different actions, simulating the fully Bayesian model for a large number of actions becomes computationally prohibitive. As a proof-of-concept, we instead conduct a brute-force simulation over ten randomly sampled values of $u_a \in (0, 1000)$, but constrain them to include at least two values close to the reference point. For each iteration, we first draw a noisy signal about each u_a, from which the DM creates a posterior distribution $u_a|s^a \sim \mathcal{N}(\lambda s^a, \sigma_x^2)$ (where $\sigma_\epsilon = \sigma_p = \tilde{u}_a = 0$). We truncate the posterior distributions on $u_r = 500 \gtrless u_a$. We then sample a candidate utility path from the sequence of truncated posterior distributions, and accept this path only if it is strictly monotonic across all outcomes. This process is repeated until a sufficient number of valid monotonic paths are collected, and their average forms our "fully Bayesian" benchmark. Figure 11 plots this path along with our quasi-Bayesian model.

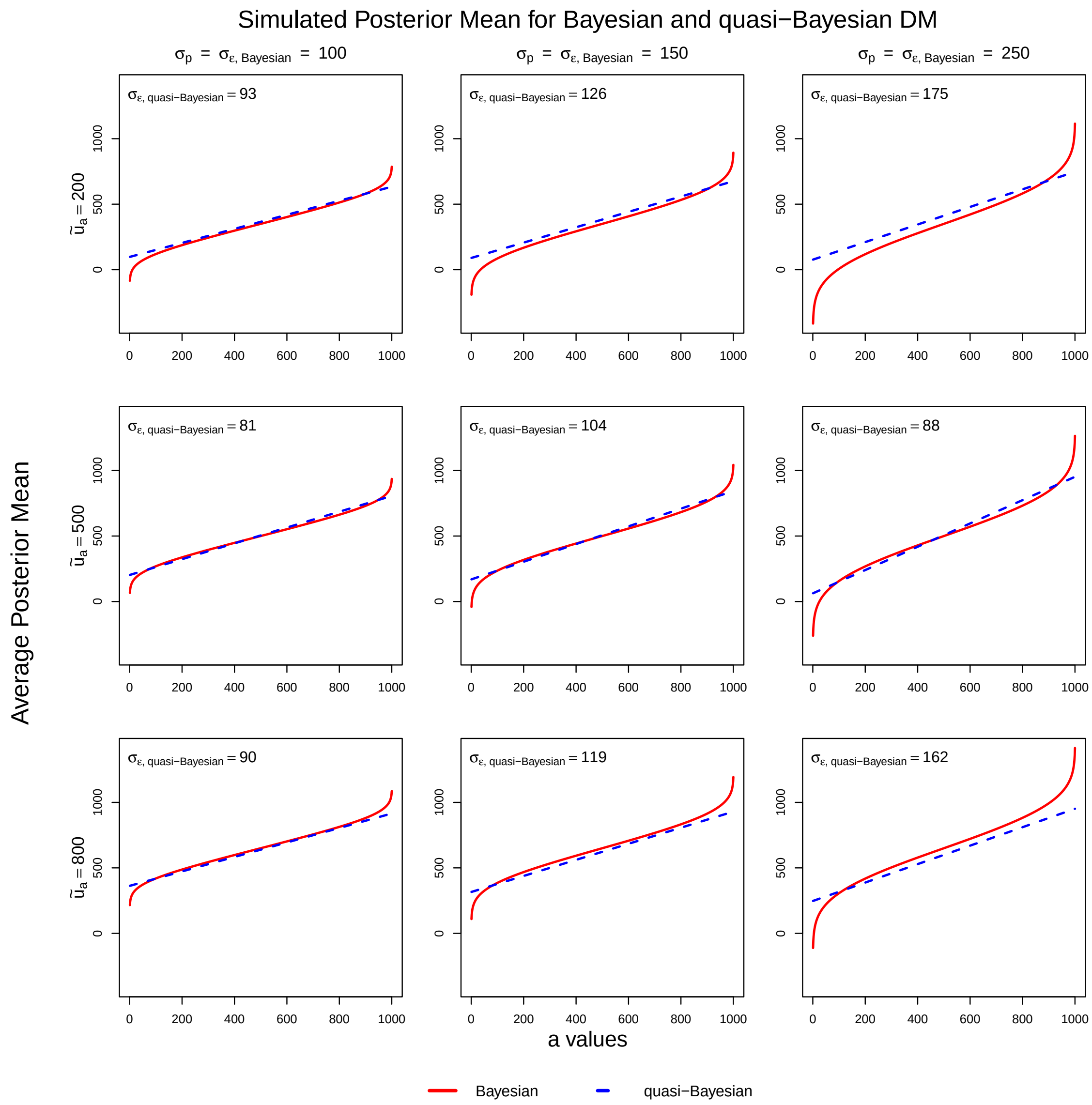

Figure 10: Simulated posterior mean utilities for a fully Bayesian agent enforcing monotonicity (solid line) and a quasi-Bayesian agent using the single-error approximation (dashed line). Each panel reports averages across 200 simulation iterations. Rows differ by prior mean $\tilde{u}_a$ (left label), and columns differ by the prior standard deviation, set equal to the Bayesian agent's signal standard deviation (top label). For each panel, the quasi-Bayesian signal standard deviation minimizing the MSE relative to the Bayesian posterior mean is indicated in the top-left corner. The continuous action variable a ranges from 1 to 1000 with unit increments for simulation purposes.

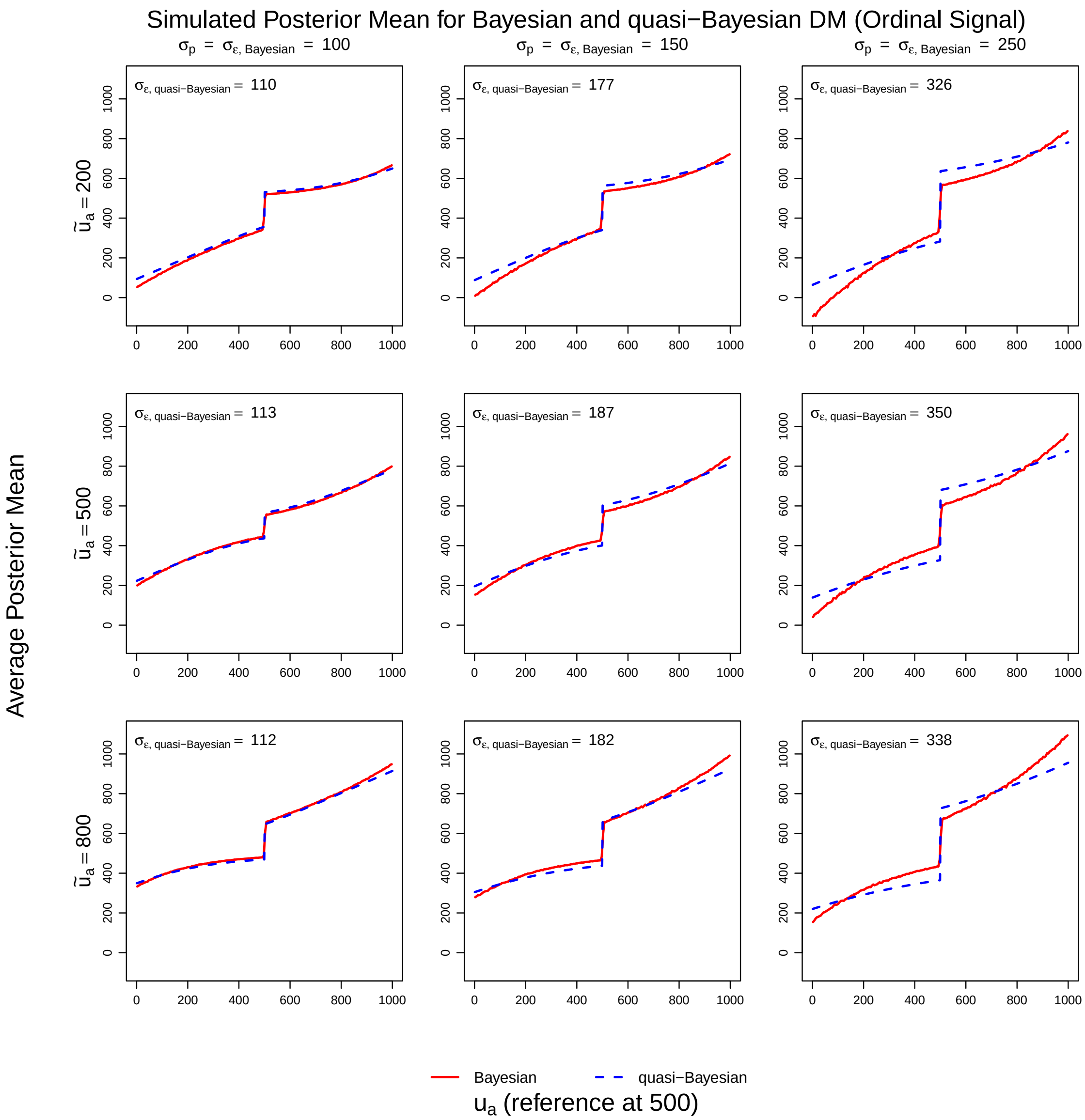

Figure 11: Comparison of posterior means for a reference-dependent DM. The solid line represents the Bayesian benchmark, while the dashed line shows the quasi-Bayesian model. Each panel reports the average of 50,000 accepted monotonic sample paths, generated via a brute-force acceptance-rejection method with posteriors truncated at the reference point ($u_r = 500$ across all panels). The quasi-Bayesian approximation is computed as the average over 1,000 draws of signal error $\epsilon_0 \sim \mathcal{N}(0, \sigma_\epsilon^2)$, where the quasi-Bayesian signal standard deviation $\sigma_{\epsilon,\text{quasi-Bayesian}}$ that minimizes the MSE relative to the Bayesian posterior mean is indicated in the top-left corner. Rows differ by prior mean $\tilde{u}_a$ (left label), and columns differ by the prior standard deviation, set equal to the Bayesian agent's signal standard deviation.

G Figures

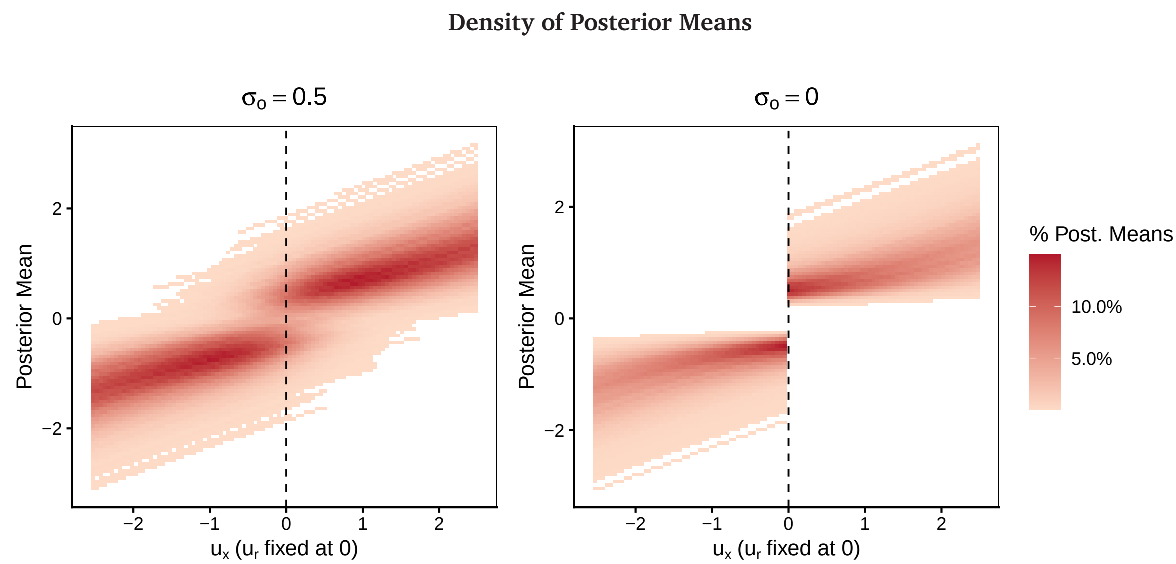

Figure 12: Distribution of posterior means $\mathbb{E}[u_x \mid s, o]$ for $\tilde{u}_x = u_r = 0$, and $\sigma_\epsilon = \sigma_p = 1$. The left panel shows the case in which the ordinal signal is noisy, $\sigma_o = 0.5$. The right panel shows the case the ordinal signal is noiseless, $\sigma_o = 0$. Each panel shows, for 100 steps of $u_x \in [-2.5, 2.5]$, the average over 10,000 draws of $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$ and $\nu \sim \mathcal{N}(0, \sigma_o^2)$.

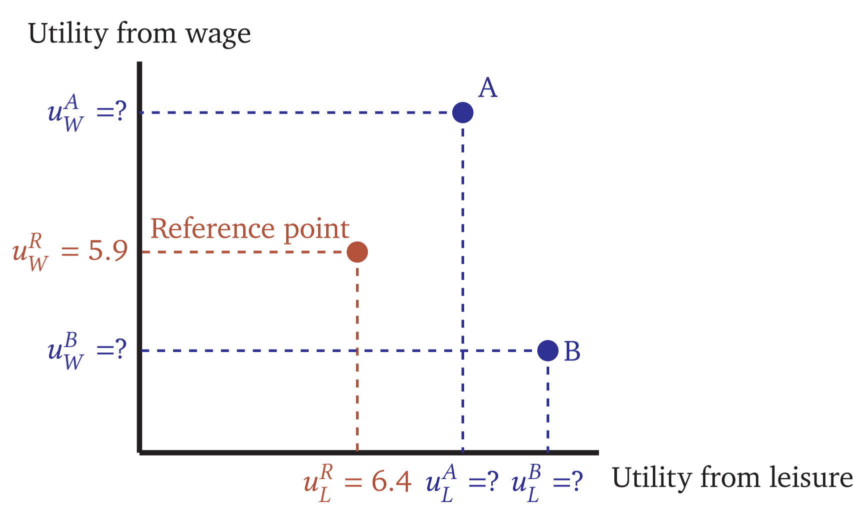

Figure 13: Illustration of the asymmetric dominance or improvements-vs.-tradeoffs effect

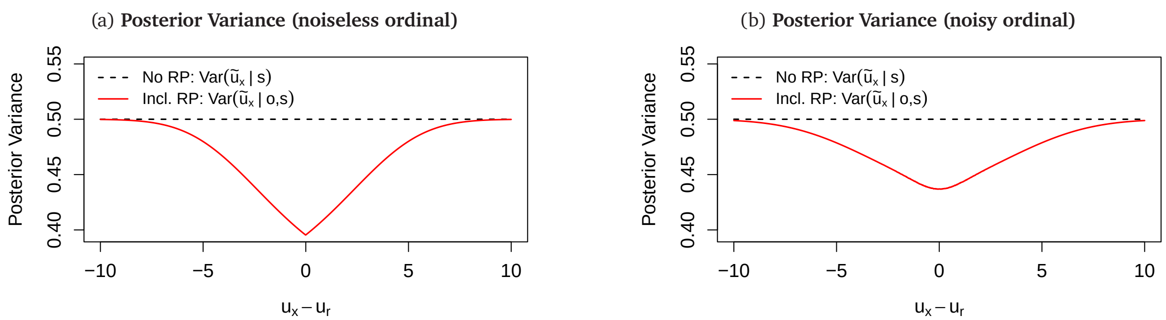

Figure 14: Posterior variance with and without ordinal information. Both panels show the reference-free posterior variance (black) for comparison. Parameters are held fixed across panels with $\sigma_p, \sigma_e, \sigma_r = 1$; the left panel uses $\sigma_o = 0$ (noiseless), while the right panel uses $\sigma_o = 1$ (noisy ordinal).